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SYMBOLIC SOLUTION OF A MULTILAYER OCEAN WAVEGUIDE PROBLEM WITH ARBITRARY DEPTH DEPENDENT AMBIENT DENSITY AND SOUND SPEED PROFILES

by

LCDR Charles Joseph Young, Jr., USN

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Thesis Advisor: Lawrence J. Ziomek

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The main purpose of this thesis was to obtain the symbolic solution of a multilayer (four fluid media) ocean waveguide problem. The waveguide was assumed to have depth-dependent ambient density and sound speed profiles in all fluid media, and arbitrarily shaped boundaries between all fluid media. A system of 28 equations in 17 unknowns was generated by satisfying all of the boundary conditions (including the boundary condition at the source) in cylindrical coordinates. The problem was set up as a weighted least squares problem for symbolic solution by the computer program Mathematica. Due to software and hardware constraints, a symbolic solution for the most general case was not obtained. However, by making all of the boundaries plane, parallel boundaries, two cases were successfully programmed, yielding symbolic solutions which were verified by comparison to previously known results.

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Symbolic Solution of a Multilayer Ocean Waveguide Problem With Arbitrary Depth Dependent Ambient Density and Sound Speed Profiles

by

Charles Joseph Young, Jr.
Lieutenant Commander, United States Navy
B.S., New Jersey Institute of Technology, 1978

Submitted in partial fulfillment of the
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December 1991

Author: Charles Joseph Young, Jr.
Charles Joseph Young, Jr.

Approved by: Lawrence J. Ziomek
Lawrence J. Ziomek, Thesis Advisor

Hung-Mou Lee
Hung-Mou Lee, Second Reader

Anthony A. Atchley
Anthony A. Atchley, Chairman, Engineering Acoustics
Academic Committee

ABSTRACT

The main purpose of this thesis was to obtain the symbolic solution of a multilayer (four fluid media) ocean waveguide problem. The waveguide was assumed to have depth-dependent ambient density and sound-speed profiles in all fluid media, and arbitrarily shaped boundaries between all fluid media. A system of 28 equations in 17 unknowns was generated by satisfying all of the boundary conditions (including the boundary condition at the source) in cylindrical coordinates. The problem was set up as a weighted least squares problem for symbolic solution by the computer program *Mathematica*. Due to software and hardware constraints, a symbolic solution for the most general case was not obtained. However, by making all of the boundaries plane, parallel boundaries, two cases were successfully programmed, yielding symbolic solutions which were verified by comparison to previously known results.

TABLE OF CONTENTS

I. A MORE GENERAL SOLUTION TO THE LINEAR, THREE-DIMENSIONAL, LOSSLESS, HOMOGENEOUS WAVE EQUATION

The primary objective of this section is to derive the solution of the linear, three-dimensional, lossless, homogeneous wave equation (1) in the cylindrical coordinate system defined in Figure 1 for an arbitrary sound speed profile (a function of the depth coordinate, y, only).

$$\nabla^2 \varphi(t,\mathbf{r}) - \frac{1}{c^2(y)} \frac{\partial^2 \varphi(t,\mathbf{r})}{\partial t^2} = 0 \tag{1}$$

The Laplacian expressed in the defined cylindrical coordinate system is given by (2).

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial^2 \phi} + \frac{\partial^2}{\partial y^2} \tag{2}$$

The remaining quantities of (1) are defined as follows:

- $\varphi(t,\mathbf{r})$ is the velocity potential at time t at a position $\mathbf{r} = (r,\phi,y)$ expressed in units of square meters per second, and
- $c(y)$ is the arbitrary, depth-dependent speed of sound expressed in units of meters per second.

The first assumption we will make in this derivation is that the source of acoustic energy in the waveguide has a time-harmonic dependence. This assumption may be justified by the facts that any arbitrary time dependence can be expressed as a summation of time-harmonic terms using Fourier

analysis. In addition, one of the basic tenets of linear acoustics allows the velocity potential to be expressed as a linear combination (using the superposition principle) of such terms.

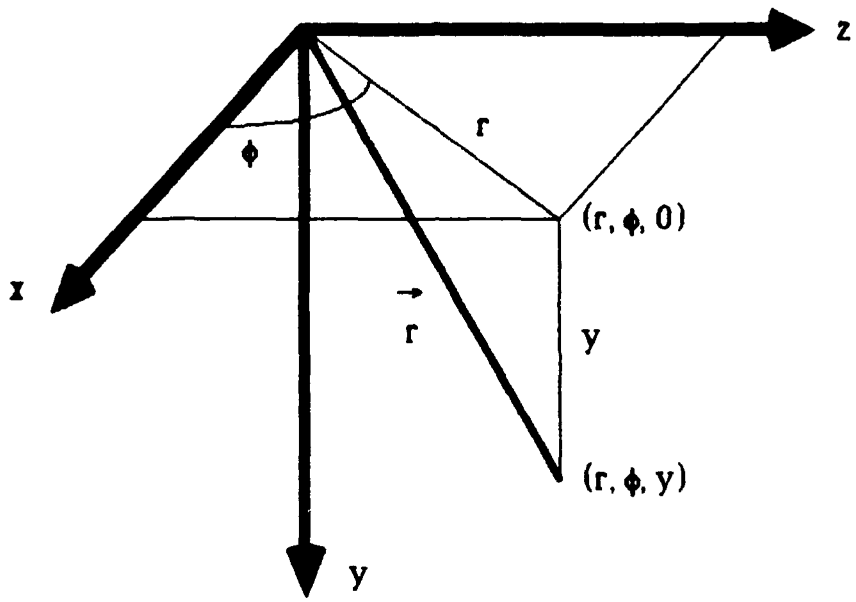

Figure 1. The Cylindrical Coordinate System

Therefore, the resulting velocity potential will also have a time-harmonic dependence given by

$$\varphi(t,\mathbf{r}) = \varphi(\mathbf{r})\, e^{j2\pi ft}\,, \tag{3}$$

where, f is the source frequency in Hertz.

Substituting (3) into (1) gives

$$\nabla^2[\varphi(\mathbf{r})\, e^{j2\pi ft}] - \frac{1}{c^2(y)} \frac{\partial^2[\varphi(\mathbf{r})\ e^{j2\pi ft}]}{\partial t^2} = 0\,. \tag{4}$$

Taking the time derivatives first yields

$$\frac{\partial[\varphi(\mathbf{r})\ e^{j2\pi ft}]}{\partial t} = \varphi(\mathbf{r})\frac{d(e^{j2\pi ft})}{dt} = j2\pi f\ \varphi(\mathbf{r})\ e^{j2\pi ft}$$

$$\frac{\partial^2[\varphi(\mathbf{r})\ e^{j2\pi ft}]}{\partial t^2} = \frac{\partial}{\partial t}\left\{\frac{\partial[\varphi(\mathbf{r})\ e^{j2\pi ft}]}{\partial t}\right\} = \frac{\partial[j2\pi f\ \ \varphi(\mathbf{r})\ \ e^{j2\pi ft}]}{\partial t}$$

$$= (j2\pi f)^2\ \varphi(\mathbf{r})\ e^{j2\pi ft}$$

$$\frac{\partial^2[\varphi(\mathbf{r})\ e^{j2\pi ft}]}{\partial t^2} = -\ \omega^2\ \varphi(\mathbf{r})\ e^{j2\pi ft}\ , \tag{5}$$

where $\omega = 2\pi f$ is the angular source frequency in radians per second.

Substituting (5) into (4) and observing that the Laplacian does not operate on the time-harmonic term $e^{j2\pi ft}$ yields

$$e^{j2\pi ft}\ \nabla^2\varphi(\mathbf{r}) + \frac{\omega^2}{c^2(y)}\varphi(\mathbf{r})\ e^{j2\pi ft} = 0\ .$$

Dividing out the complex exponential terms reveals the time-independent lossless Helmholtz equation

$$\nabla^2\varphi(\mathbf{r}) + k^2(y)\ \varphi(\mathbf{r}) = 0\ , \tag{6}$$

where,

$$k(y) = \frac{\omega}{c(y)} = \frac{2\pi f}{c(y)} = \frac{2\pi}{\lambda(y)} \tag{7}$$

is the wave number expressed in units of radians per meter and λ is the wavelength expressed in meters.

The next step is to find the solution to the lossless Helmholtz equation (6). This will be accomplished using the method of separation of variables. We will assume that the solution for $\varphi(\mathbf{r}) = \varphi(r,\phi,y)$ has the form

$$\varphi(\mathbf{r}) = \varphi(r,\phi,y) = R(r)\,\Phi(\phi)\,Y(y)\,. \tag{8}$$

Substituting (8) into (6) yields

$$\nabla^2[R(r)\,\Phi(\phi)\,Y(y)] + k^2(y)\,[R(r)\,\Phi(\phi)\,Y(y)] = 0\,.$$

Performing the Laplacian operation reveals

$$\frac{\partial^2[R(r)\ \Phi(\phi)\ Y(y)]}{\partial r^2} + \frac{1}{r}\frac{\partial[R(r)\ \Phi(\phi)\ Y(y)]}{\partial r} + \frac{1}{r^2}\frac{\partial^2[R(r)\ \Phi(\phi)\ Y(y)]}{\partial \phi^2}$$

$$+ \frac{\partial^2[R(r)\ \Phi(\phi)\ Y(y)]}{\partial y^2} + k^2(y)\,[R(r)\,\Phi(\phi)\,Y(y)] = 0\,. \tag{9}$$

The partial derivatives of (9) may be replaced with total derivatives since the functions R, Φ, and Y are each simply functions of the single variables r, ϕ, and y respectively. Continuing, (9) may be written as

$$\Phi(\phi)\,Y(y)\,\frac{d^2R(r)}{dr^2} + \Phi(\phi)\,Y(y)\,\frac{1}{r}\,\frac{dR(r)}{dr} + \frac{1}{r^2}\,R(r)\,Y(y)\,\frac{d^2\Phi(\phi)}{d\phi^2}$$

$$+ R(r)\,\Phi(\phi)\,\frac{d^2Y(y)}{dy^2} + k^2(y)\,[R(r)\,\Phi(\phi)\,Y(y)] = 0\,. \tag{10}$$

Dividing (10) by the product $[R(r)\,\Phi(\phi)\,Y(y)]$ yields

$$\frac{1}{R(r)}\,\frac{d^2R(r)}{dr^2} + \frac{1}{rR(r)}\,\frac{dR(r)}{dr} + \frac{1}{r^2\Phi(\phi)}\,\frac{d^2\Phi(\phi)}{d\phi^2} + \frac{1}{Y(y)}\,\frac{d^2Y(y)}{dy^2} + k^2(y) = 0\,.$$

Separating out the depth dependence reveals

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + k^2(y) = -\frac{1}{R(r)}\frac{d^2R(r)}{dr^2} - \frac{1}{rR(r)}\frac{dR(r)}{dr} - \frac{1}{r^2\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} . \qquad (11)$$

Since the left-hand side of (11) is a function of a single variable (y) and the right-hand side is a function of two variables (r, ϕ), the equality can only be true if each side is equal to the same constant, say k_r^2. Thus, the right-hand side of (11) becomes

$$-\frac{1}{R(r)}\frac{d^2R(r)}{dr^2} - \frac{1}{rR(r)}\frac{dR(r)}{dr} - \frac{1}{r^2\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = k_r^2 .$$

Multiplying through by r^2 and rearranging reveals

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + k_r^2 r^2 = -\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} . \qquad (12)$$

Since the left-hand side (LHS) of (12) is a function of a single variable (r) and the right-hand side (RHS) is a function of a different single variable (ϕ), the equality can hold only if each side is equal to the same constant, say n^2. Thus, the right-hand side of (12) becomes

$$-\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = n^2 . \qquad (13)$$

Multiplying (13) through by $-\Phi(\phi)$ and rearranging yields

$$\frac{d^2\Phi(\phi)}{d\phi^2} + n^2\,\Phi(\phi) = 0 . \qquad (14)$$

Equation (14) is a second order ordinary differential equation with the following exact solution:

$$\Phi(\phi) = A_\phi \cos(n\phi) + B_\phi \sin(n\phi)\,, \tag{15}$$

where A_ϕ and B_ϕ are in general complex constants whose values are determined by satisfying boundary conditions.

Now, the left-hand side of (12) must be evaluated. Since the left-hand side of (12) must also be equal to n^2, we have

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + k_r^2\, r^2 = n^2\,. \tag{16}$$

Multiplying (16) by $\frac{R(r)}{r^2}$ and rearranging reveals

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left[k_r^2 - \frac{n^2}{r^2}\right] R(r) = 0\,. \tag{17}$$

Let

$$R(r) = g(k_r\, r)\,. \tag{18}$$

Then

$$\frac{dR(r)}{dr} = \frac{d[g(k_r\, r)]}{dr} = \frac{d[g(k_r\, r)]}{d(k_r\, r)}\frac{d(k_r\, r)}{dr}$$

or,

$$\frac{dR(r)}{dr} = k_r \frac{d[g(k_r r)]}{d(k_r r)} . \tag{19}$$

Additionally,

$$\frac{d^2R(r)}{dr^2} = \frac{d}{dr}\left[\frac{dR(r)}{dr}\right] = \frac{d}{dr}\left[k_r \frac{d\{g(k_r r)\}}{d(k_r r)}\right] = k_r \frac{d}{d(k_r r)}\left[\frac{d\{g(k_r r)\}}{dr}\right]$$

or,

$$\frac{d^2R(r)}{dr^2} = k_r^2 \frac{d^2[g(k_r r)]}{d(k_r r)^2} . \tag{20}$$

Substituting (18) through (20) into (17) yields

$$k_r^2 \frac{d^2[g(k_r r)]}{d(k_r r)^2} + \frac{k_r}{r} \frac{d[g(k_r r)]}{d(k_r r)} + \left[k_r^2 - \frac{n^2}{r^2}\right] g(k_r r) = 0 . \tag{21}$$

Dividing (21) by k_r^2 yields

$$\frac{d^2[g(k_r r)]}{d(k_r r)^2} + \frac{1}{k_r r} \frac{d[g(k_r r)]}{d(k_r r)} + \left[1 - \frac{n^2}{(k_r r)^2}\right] g(k_r r) = 0 . \tag{22}$$

Equation (22) is known as Bessel's differential equation, which has the following exact solution:

$$g(k_r r) = A_r J_n(k_r r) + B_r N_n(k_r r) , \tag{23}$$

where J_n and N_n are the Bessel functions of the first and second kind respectively, and A_r and B_r are in general complex constants whose values are determined by satisfying boundary conditions.

Since we let $R(r) = g(k_r r)$, the exact solution for $R(r)$ can be written as follows:

$$R(r) = A_r J_n(k_r r) + B_r N_n(k_r r) . \tag{24}$$

Now, the left-hand side of (11) must be set equal to k_r^2, revealing

$$\frac{1}{Y(y)} \frac{d^2Y(y)}{dy^2} + k^2(y) = k_r^2 . \tag{25}$$

Multiplying (25) by $Y(y)$ and rearranging yields

$$\frac{d^2Y(y)}{dy^2} + \left[k^2(y) - k_r^2\right] Y(y) = 0 . \tag{26}$$

Let

$$k_y^2(y) = k^2(y) - k_r^2 . \tag{27}$$

Using (27) to rewrite (26) yields

$$\frac{d^2Y(y)}{dy^2} + k_y^2(y)\, Y(y) = 0 . \tag{28}$$

Since the coefficient $k_y^2(y)$ in (28) is an arbitrary function of depth, y, an exact solution to this differential equation cannot be determined.

In order to continue this derivation of a generalized solution to the wave equation, we will assume that the solution to (28) can be determined by other means, and is given simply by

$$Y(y) = Y(y) . \tag{29}$$

In certain cases, the sound speed profile, c(y), may be a function such that an exact solution to (28) may be found (say by consulting tables). In other specific cases, approximations can be made in order to put (28) in a form whereby a known solution may be found. However, in the most general case, (28) will have to be evaluated numerically.

Recall that (8) specifies the velocity potential as

$$\varphi(r,\phi,y) = R(r)\,\Phi(\phi)\,Y(y) . \tag{30}$$

Substituting (15), (24), and (29) into (30) yields the following general solution for the velocity potential:

$$\varphi(r,\phi,y) = \left[A_r\,J_n(k_r\,r) + B_r\,N_n(k_r\,r)\right]\left[A_\phi\cos(n\phi) + B_\phi\sin(n\phi)\right]Y(y) . \tag{31}$$

Finally, recall that (3) described the velocity potential as a function of time and position as

$$\varphi(t,\mathbf{r}) = \varphi(\mathbf{r})\,e^{j2\pi ft} . \tag{32}$$

Therefore, the complete general solution for the velocity potential is given by

$$\varphi(t,r,\phi,y) = \left[A_r\,J_n(k_r\,r) + B_r\,N_n(k_r\,r)\right]\left[A_\phi\cos(n\phi) + B_\phi\sin(n\phi)\right]Y(y)\,e^{j2\pi ft} . \tag{33}$$

II. DESCRIPTION OF THE GENERALIZED OCEAN WAVEGUIDE MODEL

The next logical step in this analysis is to apply the solution to the linear, three-dimensional, lossless, homogeneous wave equation developed in the previous section to the generalized ocean waveguide model. Before continuing with the mathematical derivation, a short description of the generalized ocean waveguide model chosen for analysis will be presented.

As shown in Figure 2, space has been separated into four distinct media. Medium I (which may represent the air) is completely characterized by its density ($\rho_1(y)$) and sound speed profile ($c_1(y)$), which may both be, in general, arbitrary functions of depth coordinate, y, only. Medium II (which may represent the ocean water) is completely characterized by its density ($\rho_2(y)$) and sound speed profile ($c_2(y)$), which may both be, in general, arbitrary functions of depth coordinate, y, only. Medium III (which may represent the upper layer (sediment) of the ocean bottom) is completely characterized by its density ($\rho_3(y)$) and sound speed profile ($c_3(y)$), which may both be, in general, arbitrary functions of depth coordinate, y, only. Medium IV (which may represent a second fluid layer of the ocean bottom) is completely characterized by its density ($\rho_4(y)$) and sound speed profile ($c_4(y)$), which may both be, in general, arbitrary functions of depth coordinate, y, only.

Media I and II are separated by a boundary (which may represent the ocean surface) at a depth of $y_S(r,\phi)$ meters, where y_S may in general be an

arbitrary function of horizontal range, r, and azimuthal angle ϕ. Media II and III are separated by a boundary (which may represent the ocean bottom) at a depth of $y_{B_1}(r,\phi)$ meters, where y_{B_1} may in general be an arbitrary function of horizontal range, r, and azimuthal angle ϕ. Media III and IV are separated by a boundary (which may represent the interface between two different layers in the bottom) at a depth of $y_{B_2}(r,\phi)$ meters, where y_{B_2} may in general be an arbitrary function of horizontal range, r, and azimuthal angle ϕ.

Medium II requires some additional examination. First, medium II contains the omnidirectional, time-harmonic point source located at a fixed depth of y_0 meters and at a fixed horizontal range of zero meters. Medium II is also further separated into two distinct subregions, labelled medium IIa and medium IIb, by an artificial boundary located at the depth of the source. This additional boundary is required to satisfy the source boundary conditions. The density and sound speed profiles for these subregions are completely defined by the generic medium II descriptions previously discussed.

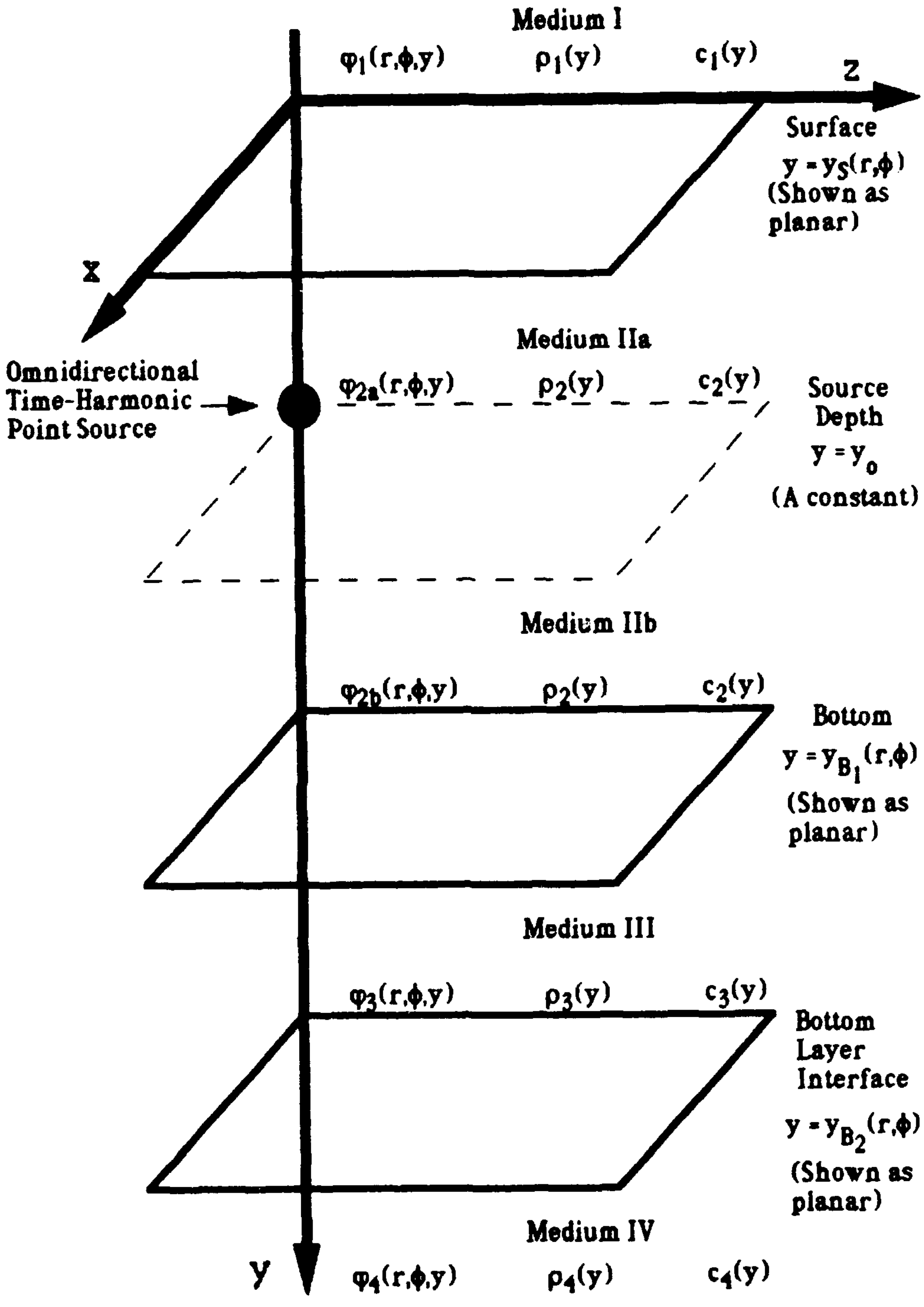

Figure 2. Generalized Ocean Waveguide Model
(Based on Ziomek (1991, Figure 3.9-1))

III. DEVELOPMENT OF THE GENERAL WAVEGUIDE MODEL SOLUTION

The primary objective of this section is to derive general expressions for the velocity potentials in media IIa and IIb based on the wave equation solution previously derived and the appropriate general boundary conditions.

The next assumption required in this derivation is that the source of acoustic energy in the waveguide is an omnidirectional point source which is surrounded by the medium it is to excite acoustically. This assumption may be justified by the fact that the velocity potential field generated by an arbitrary acoustic source array may be expressed as the summation of the velocity potential fields generated by the individual omnidirectional sources which make up the array. Assuming that the source is surrounded by the medium it is to drive acoustically is reasonable for many practical acoustic systems.

Recall that the velocity potential is given by

$$\varphi(t,r,\phi,y) = \left[A_r\, J_n(k_r\, r) + B_r\, N_n(k_r\, r)\right]\left[A_\phi \cos(n\phi) + B_\phi \sin(n\phi)\right] Y(y)\, e^{j2\pi ft}\,. \quad (34)$$

The omnidirectional point source assumption implies that the source radiates acoustic energy equally well in all directions. However, since we have assumed an arbitrarily varying surface and an arbitrarily varying bottom for this waveguide, this assumption *does not* imply that the velocity potential field lacks an angular dependence (that is, in general, the velocity

potential at a given range and depth will depend on the azimuthal angle because of the differences in the interactions of the sound rays with the arbitrary surface and bottom). Assuming that the source is surrounded by the medium (and since the source is located on the y-axis, this assumption implies that the y-axis is surrounded by the medium) implies that the arbitrary constant n must be an integer in order to ensure that the velocity potential is single-valued for azimuthal angles ϕ in excess of 2π radians (i.e., the velocity potential must be a periodic function with period 2π radians).

Since the velocity potential must be evaluated at the source (i.e., at r = 0), the arbitrary constant B_r must be set equal to zero in order to eliminate the Neumann function solution (since all of the Neumann functions tend to infinity as r tends to zero) (see Boas (1983, p. 513 and p. 525) for further proof). Thus, the general solution is reduced to

$$\varphi(t,r,\phi,y) = A_r\, J_n(k_r\, r) \left[A_\phi \cos(n\phi) + B_\phi \sin(n\phi)\right] Y(y)\, e^{j2\pi ft} . \qquad (35)$$

A. VELOCITY POTENTIAL IN MEDIUM I

Based on (35), the velocity potential in medium I can be expressed as

$$\varphi_I(t,r,\phi,y) = A_{r_I}\, J_n(k_{r_I}\, r) \left[A_{\phi_I} \cos(n\phi) + B_{\phi_I} \sin(n\phi)\right] Y_I^-(y)\, e^{j2\pi ft} \qquad (36)$$

where $Y_I^-(y)$ represents the solution to (28) for the traveling wave in the negative y direction.

Carrying out the indicated multiplications, and defining the following new constants:

$$A_{r_I} A_{\phi_I} = B_I$$

$$A_{r_I} B_{\phi_I} = A_I ,$$

reveals the general form for the velocity potential in medium I

$$\varphi_I(t,r,\phi,y) = \left[B_I \cos(n\phi) + A_I \sin(n\phi)\right] J_n(k_{r_I} r)\, Y_I^{-}(y)\, e^{i2\pi ft} , \qquad (37)$$

where A_I and B_I are arbitrary complex constants whose values will be determined by satisfying boundary conditions.

Additionally,

$$k_I^2(y) = k_{r_I}^2 + k_{y_I}^2(y) = \frac{(2\pi f)^2}{c_I^2(y)} . \qquad (38)$$

where $k_I(y)$ is the wave number in medium I, k_{r_I} is the constant radial component of the propagation vector in medium I, $k_{y_I}(y)$ is the depth component of the propagation vector in medium I, and $c_I(y)$ is the depth-dependent sound speed in medium I.

It is worth noting that there is not a solution of the form $Y_1^+(y)$ in (37). This is due to the fact that energy is considered to propagate in the negative y direction out to negative infinity without reflection.

B. VELOCITY POTENTIAL IN MEDIUM II

It is clear from the configuration that the velocity potentials in media IIa and IIb will be combinations of "incident" waves traveling toward the respective boundaries and "reflected" waves traveling away from these boundaries. With this in mind, the velocity potential in medium IIa may be written as (see equation (35))

$$\varphi_{2a}(t,r,\phi,y) = A_{r_{2a}} J_n(k_{r_2} r) \left\{ A_{\phi_{2a}} \cos n\phi + B_{\phi_{2a}} \sin n\phi \right\}$$
$$\times \left[A_{y_{2a}} Y_{2a}^+(y) + B_{y_{2a}} Y_{2a}^-(y) \right] e^{i2\pi ft}, \qquad (39)$$

where $Y_{2a}^-(y)$ represents the solution to (28) for the traveling wave in the negative y direction (incident on the boundary at $y = y_S$) and $Y_{2a}^+(y)$ represents the solution to (28) for the traveling wave in the positive y direction (reflected from the boundary at $y = y_S$).

Carrying out the indicated multiplications and defining the following new constants:

$$A_{r_{2a}} A_{\phi_{2a}} A_{y_{2a}} \equiv A_{2a}$$

$$A_{r_{2a}} A_{\phi_{2a}} B_{y_{2a}} = B_{2a}$$

$$A_{r_{2a}} B_{\phi_{2a}} A_{y_{2a}} = C_{2a}$$

$$A_{r_{2a}} B_{\phi_{2a}} B_{y_{2a}} = D_{2a} ,$$

reveals the general form for the velocity potential in medium IIa

$$\varphi_{2a}(t,r,\phi,y) = \Big\{A_{2a} \cos n\phi\, Y^{+}_{2a}(y) + B_{2a} \cos n\phi\, Y^{-}_{2a}(y)$$

$$+ C_{2a} \sin n\phi\, Y^{+}_{2a}(y) + D_{2a} \sin n\phi\, Y^{-}_{2a}(y)\Big\}\, J_n(k_{r_2} r)\, e^{j2\pi ft} , \qquad (40)$$

where A_{2a}, B_{2a}, C_{2a}, and D_{2a} are arbitrary complex constants whose values will be determined by satisfying boundary conditions.

Similarly, the velocity potential in medium IIb is given by

$$\varphi_{2b}(t,r,\phi,y) = \Big\{A_{2b} \cos n\phi\, Y^{+}_{2b}(y) + B_{2b} \cos n\phi\, Y^{-}_{2b}(y)$$

$$+ C_{2b} \sin n\phi\, Y^{+}_{2b}(y) + D_{2b} \sin n\phi\, Y^{-}_{2b}(y)\Big\}\, J_n(k_{r_2} r)\, e^{j2\pi ft} , \qquad (41)$$

where $Y^{+}_{2b}(y)$ represents the solution to (28) for the traveling wave in the positive y direction (incident on the boundary at $y = y_{B_1}$), $Y^{-}_{2b}(y)$ represents the solution to (28) for the traveling wave in the negative y direction

(reflected from the boundary at $y = y_{B_1}$), and A_{2b}, B_{2b}, C_{2b}, and D_{2b} are arbitrary complex constants whose values will be determined by satisfying boundary conditions.

Additionally,

$$k_2^2(y) = k_{r_2}^2 + k_{y_2}^2(y) = \frac{(2\pi f)^2}{c_2^2(y)}, \tag{42}$$

where $k_2(y)$ is the wave number in medium II, k_{r_2} is the constant radial component of the propagation vector in medium II, $k_{y_2}(y)$ is the depth component of the propagation vector in medium II, and $c_2(y)$ is the depth-dependent sound speed in medium II.

C. VELOCITY POTENTIAL IN MEDIUM III

It is again clear from the configuration of our waveguide that the velocity potential in medium III will be a combination of a traveling wave in the positive y direction and a traveling wave in the negative y direction. Thus, the velocity potential in medium III can be derived using the techniques presented previously (for medium II). Performing this analysis yields

$$\begin{aligned} \varphi_3(t,r,\phi,y) = \Big\{ & A_3 \cos n\phi \, Y_3^+(y) + B_3 \cos n\phi \, Y_3^-(y) \\ & + C_3 \sin n\phi \, Y_3^+(y) + D_3 \sin n\phi \, Y_3^-(y) \Big\} J_n(k_{r_3} r) \, e^{i2\pi f t}, \end{aligned} \tag{43}$$

where $Y_3^+(y)$ represents the solution to (28) for the traveling wave in the positive y direction (incident on the boundary at $y = y_{B_2}$), $Y_3^-(y)$ represents the solution to (28) for the traveling wave in the negative y direction (reflected from the boundary at $y = y_{B_2}$), and A_3, B_3, C_3, and D_3 are arbitrary complex constants whose values will be determined by satisfying boundary conditions.

Additionally,

$$k_3^2(y) = k_{r_3}^2 + k_{y_3}^2(y) = \frac{(2\pi f)^2}{c_3^2(y)}, \qquad (44)$$

where $k_3(y)$ is the wave number in medium III, k_{r_3} is the constant radial component of the propagation vector in medium III, $k_{y_3}(y)$ is the depth component of the propagation vector in medium III, and $c_3(y)$ is the depth-dependent sound speed in medium III.

D. VELOCITY POTENTIAL IN MEDIUM IV

Finally, in medium IV, the velocity potential is given by (see (35))

$$\varphi_4(t,r,\phi,y) = A_{r_4} J_n(k_{r_4} r) \left\{ A_{\phi_4} \cos n\phi + B_{\phi_4} \sin n\phi \right\} Y_4^+(y)\, e^{j2\pi ft}, \qquad (45)$$

where $Y_4^+(y)$ represents the solution to (28) for the traveling wave in the positive y direction.

Carrying out the indicated multiplications and defining the following new constants:

$$A_{r_4} A_{\phi_4} = A_4$$

$$A_{r_4} B_{\phi_4} = B_4 ,$$

reveals the general form for the velocity potential in medium IV

$$\varphi_4(t,r,\phi,y) = \left\{A_4 \cos n\phi + B_4 \sin n\phi\right\} J_n(k_{r_4} r)\, Y_4^+(y)\, e^{j2\pi ft} , \qquad (46)$$

where A_4 and B_4 are arbitrary complex constants whose values will be determined by satisfying boundary conditions.

Additionally,

$$k_4^2(y) = k_{r_4}^2 + k_{y_4}^2(y) = \frac{(2\pi f)^2}{c_4^2(y)} , \qquad (47)$$

where $k_4(y)$ is the wave number in medium IV, k_{r_4} is the constant radial component of the propagation vector in medium IV, $k_{y_4}(y)$ is the depth

component of the propagation vector in medium IV, and $c_4(y)$ is the depth-dependent sound speed in medium IV.

It is worth noting that there is not a solution of the form $Y_4^-(y)$ in (46). This is due to the fact that energy is considered to propagate in the positive y direction out to positive infinity without reflection.

E. BOUNDARY CONDITIONS

There are three different types of boundary conditions that must be applied to the solution of our problem. The first is the condition of "continuity of acoustic pressure" across a boundary. This condition requires that the acoustic pressure evaluated at a particular spatial location and time on one side of the boundary be identically equal to the pressure evaluated at the same spatial location and time on the other side of the boundary. Kinsler (1982, p. 125) describes this condition as meaning that there is no net force acting on the massless boundary separating the two media.

The second type of boundary condition is that of continuity of the normal component of the acoustic particle velocity across a boundary. This condition requires that the normal component of the acoustic particle velocity evaluated at a particular spatial location and time on one side of the boundary be identically equal to the normal component of the acoustic particle velocity evaluated at the same spatial location and time on the other side of the boundary. Kinsler (1982, p. 126) describes this condition as meaning that the media remain in contact with each other.

The final type of boundary condition is that of discontinuity of the normal component of the acoustic particle velocity across the boundary at the depth of the source. Officer (1958, p. 124 and following) and Ziomek (1991, discussion following equation (3.9-37)) describe this condition as being required to ensure that the solution to the wave equation reduces to that of an omnidirectional point source when the boundaries at $y = y_S$, $y = y_{B_1}$, and $y = y_{B_2}$ are removed.

In developing the required boundary condition expressions, we will consider the various types of conditions in the order previously discussed. Thus, the first boundary condition to be applied to this problem is that of continuity of acoustic pressure across the boundary at $y = y_S$. This implies:

$$p_1(t,r,\phi,y_S) = p_{2a}(t,r,\phi,y_S) .$$

In general (from the fluid dynamics derivation), the acoustic pressure can be related to the velocity potential by the following:

$$p(t,r,\phi,y) = -\rho_0(r,\phi,y)\frac{\partial\varphi(t,r,\phi,y)}{\partial t}, \tag{48}$$

where $\rho_0(r,\phi,y)$, representing the ambient (equilibrium) density of the medium, is, in general, an arbitrary function of the spatial variables r, ϕ, y, but is not a function of time.

Since we have assumed that density is simply a function of depth coordinate y, and assumed a time-harmonic source, the acoustic pressure may be written as

$$p(t,r,\phi,y) = -j\,\omega\,\rho_0(y)\,\varphi(t,r,\phi,y)\,. \tag{49}$$

Thus, the acoustic pressures in the various media may be expressed as

$$p_1(t,r,\phi,y) = -j\,\omega\,\rho_1(y)\,\varphi_1(t,r,\phi,y)\,, \tag{50}$$

$$p_{2a}(t,r,\phi,y) = -j\,\omega\,\rho_2(y)\,\varphi_{2a}(t,r,\phi,y)\,, \tag{51}$$

$$p_{2b}(t,r,\phi,y) = -j\,\omega\,\rho_2(y)\,\varphi_{2b}(t,r,\phi,y)\,, \tag{52}$$

$$p_3(t,r,\phi,y) = -j\,\omega\,\rho_3(y)\,\varphi_3(t,r,\phi,y)\,, \tag{53}$$

and,

$$p_4(t,r,\phi,y) = -j\,\omega\,\rho_4(y)\,\varphi_4(t,r,\phi,y)\,, \tag{54}$$

where $\rho_1(y)$, $\rho_2(y)$, $\rho_3(y)$, and $\rho_4(y)$ are the ambient densities in media I, II, III, and IV, respectively.

Returning to the specific boundary condition being examined, we set p_1 (given by (50)) equal to p_{2a} (given by (51)) at $y = y_S$, and divide out the common factor of $-j\,\omega$ revealing

$$\rho_1(y_S)\,\varphi_1(t,r,\phi,y_S) = \rho_2(y_S)\,\varphi_{2a}(t,r,\phi,y_S)\,. \tag{55}$$

Substituting the derived velocity potentials for media I (37) and IIa (40) into (55) yields

$$\rho_1(y_S)\left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} J_n(k_{r_1} r)\, Y_1^-(y_S)\, e^{j2\pi ft}$$

$$= \rho_2(y_S)\left\{A_{2a} \cos n\phi\, Y_{2a}^+(y_S) + B_{2a} \cos n\phi\, Y_{2a}^-(y_S)\right.$$

$$\left. + C_{2a} \sin n\phi\, Y_{2a}^+(y_S) + D_{2a} \sin n\phi\, Y_{2a}^-(y_S)\right\} J_n(k_{r_2} r)\, e^{j2\pi ft} . \qquad (56)$$

The time dependence is eliminated by dividing (56) through by the complex exponential term $e^{j2\pi ft}$. Carrying out the indicated multiplications and factoring yields the following:

$$\left[\rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^+(y_S)\, A_{2a} + \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^-(y_S)\, B_{2a}\right] \cos n\phi$$

$$+ \left[\rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^+(y_S)\, C_{2a} + \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^-(y_S)\, D_{2a}\right] \sin n\phi$$

$$= \rho_1(y_S)\, J_n(k_{r_1} r)\, Y_1^-(y_S)\, B_1 \cos n\phi + \rho_1(y_S)\, J_n(k_{r_1} r)\, Y_1^-(y_S)\, A_1 \sin n\phi . \quad (57)$$

Setting the coefficients of cos $n\phi$ on the left-hand side of (57) equal to the coefficients of cos $n\phi$ on the right-hand side of (57), setting the coefficients of sin $n\phi$ on the left-hand side of (57) equal to the coefficients of sin $n\phi$ on the right-hand side of (57), and rearranging the equations generated by this process reveals the pair of equations representing the first boundary condition (BC #1)

$$\rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^+(y_S)\, A_{2a} + \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^-(y_S)\, B_{2a}$$

$$- \rho_1(y_S)\, J_n(k_{r_1} r)\, Y_1^-(y_S)\, B_1 = 0 , \qquad (58)$$

and

$$\rho_2(y_S)\, J_n(k_{r_2} r)\, Y^+_{2a}(y_S)\, C_{2a} + \rho_2(y_S)\, J_n(k_{r_2} r)\, Y^-_{2a}(y_S)\, D_{2a}$$

$$- \rho_1(y_S)\, J_n(k_{r_1} r)\, Y^-_1(y_S)\, A_1 = 0 \,. \tag{59}$$

It should be noted here that (58) and (59) are valid only if the associated trigonometric function is not identically zero for all values of azimuthal angle ϕ (for instance, if $n = 0$, then $\sin n\phi$ is identically zero for all values of ϕ implying that (59) is no longer a valid boundary condition).

The second boundary condition to be applied to this problem is that of continuity of acoustic pressure across the boundary at $y = y_o$. This implies

$$p_{2a}(t,r,\phi,y_o) = p_{2b}(t,r,\phi,y_o) \,.$$

Setting p_{2a} (given by (51)) equal to p_{2b} (given by (52)) at $y = y_o$ and dividing out the common terms reveals

$$\varphi_{2a}(t,r,\phi,y_o) = \varphi_{2b}(t,r,\phi,y_o) \,. \tag{60}$$

Substituting the derived velocity potentials for media IIa (40) and IIb (41) into (60) yields

$$J_n(k_{r_2} r) \left[A_{2a} \cos n\phi\, Y^+_{2a}(y_o) + B_{2a} \cos n\phi\, Y^-_{2a}(y_o) + C_{2a} \sin n\phi\, Y^+_{2a}(y_o)\right.$$

$$\left. + D_{2a} \sin n\phi\, Y^-_{2a}(y_o)\right] e^{j2\pi ft} = J_n(k_{r_2} r) \left[A_{2b} \cos n\phi\, Y^+_{2b}(y_o)\right.$$

$$\left. + B_{2b} \cos n\phi\, Y^-_{2b}(y_o) + C_{2b} \sin n\phi\, Y^+_{2b}(y_o) + D_{2b} \sin n\phi\, Y^-_{2b}(y_o)\right] e^{j2\pi ft} \,. \tag{61}$$

Dividing out the common terms eliminates both the horizontal range and time dependences, revealing, after factoring, the following:

$$\left[Y^{+}_{2a}(y_o)\,A_{2a} + Y^{-}_{2a}(y_o)\,B_{2a}\right]\cos n\phi + \left[Y^{+}_{2a}(y_o)\,C_{2a} + Y^{-}_{2a}(y_o)\,D_{2a}\right]\sin n\phi$$

$$= \left[Y^{+}_{2b}(y_o)\,A_{2b} + Y^{-}_{2b}(y_o)\,B_{2b}\right]\cos n\phi + \left[Y^{+}_{2b}(y_o)\,C_{2b} + Y^{-}_{2b}(y_o)\,D_{2b}\right]\sin n\phi\,. \quad (62)$$

Setting the respective coefficients of cos $n\phi$ and sin $n\phi$ equal and rearranging yields the following pair of equations representing the second boundary condition (BC #2):

$$Y^{+}_{2a}(y_o)\,A_{2a} + Y^{-}_{2a}(y_o)\,B_{2a} - Y^{+}_{2b}(y_o)\,A_{2b} - Y^{-}_{2b}(y_o)\,B_{2b} = 0\,, \quad (63)$$

and

$$Y^{+}_{2a}(y_o)\,C_{2a} + Y^{-}_{2a}(y_o)\,D_{2a} - Y^{+}_{2b}(y_o)\,C_{2b} - Y^{-}_{2b}(y_o)\,D_{2b} = 0\,. \quad (64)$$

Again, (63) and (64) are valid only if the associated trigonometric function is not identically zero for all values of azimuthal angle ϕ.

The third boundary condition to be applied to this problem is that of continuity of acoustic pressure across the boundary at $y = y_{B_1}$. This implies

$$p_{2b}(t,r,\phi,y_{B_1}) = p_3(t,r,\phi,y_{B_1})\,.$$

Setting p_{2b} (given by (52)) equal to p_3 (given by (53)) at $y = y_{B_1}$, and dividing out the common factor of $-j\,\omega$ reveals

$$\rho_2(y_{B_1})\,\varphi_{2b}(t,r,\phi,y_{B_1}) = \rho_3(y_{B_1})\,\varphi_3(t,r,\phi,y_{B_1})\,. \tag{65}$$

Substituting the derived velocity potentials for media IIb (41) and III (43) into (65) yields

$$\begin{aligned}
&\rho_2(y_{B_1})\,J_n(k_{r_2}r)\Big[A_{2b}\cos n\phi\,Y^{+}_{2b}(y_{B_1}) + B_{2b}\cos n\phi\,Y^{-}_{2b}(y_{B_1}) \\
&\quad + C_{2b}\sin n\phi\,Y^{+}_{2b}(y_{B_1}) + D_{2b}\sin n\phi\,Y^{-}_{2b}(y_{B_1})\Big]\,e^{j2\pi ft} \\
&= \rho_3(y_{B_1})\,J_n(k_{r_3}r)\Big[A_3\cos n\phi\,Y^{+}_3(y_{B_1}) + B_3\cos n\phi\,Y^{-}_3(y_{B_1}) \\
&\quad + C_3\sin n\phi\,Y^{+}_3(y_{B_1}) + D_3\sin n\phi\,Y^{-}_3(y_{B_1})\Big]\,e^{j2\pi ft}\,.
\end{aligned} \tag{66}$$

The time dependence is eliminated by dividing (66) through by the complex exponential term $e^{j2\pi ft}$. Carrying out the indicated multiplications and factoring reveals the following:

$$\begin{aligned}
&\Big[\rho_2(y_{B_1})\,J_n(k_{r_2}r)\,Y^{+}_{2b}(y_{B_1})\,A_{2b} + \rho_2(y_{B_1})\,J_n(k_{r_2}r)\,Y^{-}_{2b}(y_{B_1})\,B_{2b}\Big]\cos n\phi \\
&+ \Big[\rho_2(y_{B_1})\,J_n(k_{r_2}r)\,Y^{+}_{2b}(y_{B_1})\,C_{2b} + \rho_2(y_{B_1})\,J_n(k_{r_2}r)\,Y^{-}_{2b}(y_{B_1})\,D_{2b}\Big]\sin n\phi
\end{aligned}$$

$$- \left[\rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^+(y_{B_1})\, A_3 + \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_1})\, B_3\right] \cos n\phi$$

$$+ \left[\rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^+(y_{B_1})\, C_3 + \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_1})\, D_3\right] \sin n\phi\,. \quad (67)$$

Setting the respective coefficients of cos $n\phi$ and sin $n\phi$ equal and rearranging yields the following pair of equations representing the third boundary condition (BC #3):

$$\rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^+(y_{B_1})\, A_{2b} + \rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^-(y_{B_1})\, B_{2b}$$

$$- \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^+(y_{B_1})\, A_3 - \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_1})\, B_3 = 0\,, \qquad (68)$$

and

$$\rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^+(y_{B_1})\, C_{2b} + \rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^-(y_{B_1})\, D_{2b}$$

$$- \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^+(y_{B_1})\, C_3 - \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_1})\, D_3 = 0\,. \qquad (69)$$

Again, (68) and (69) are valid only if the associated trigonometric function is not identically zero for all values of azimuthal angle ϕ.

The fourth boundary condition to be applied to this problem is that of continuity of acoustic pressure across the boundary at $y = y_{B_2}$. This implies

$$p_3(t,r,\phi,y_{B_2}) = p_4(t,r,\phi,y_{B_2})\,.$$

Setting p_3 (given by (53)) equal to p_4 (given by (54)) at $y = y_{B_2}$, and dividing out the common factor of $-j\,\omega$ reveals

$$\rho_3(y_{B_2})\,\varphi_3(t,r,\phi,y_{B_2}) = \rho_4(y_{B_2})\,\varphi_4(t,r,\phi,y_{B_2})\,. \tag{70}$$

Substituting the derived velocity potentials for media III (43) and IV (46) into (70) yields

$$\begin{aligned} &\rho_3(y_{B_2})\,J_n(k_{r_3} r)\left[A_3 \cos n\phi\, Y_3^+(y_{B_2}) + B_3 \cos n\phi\, Y_3^-(y_{B_2})\right. \\ &\left. + C_3 \sin n\phi\, Y_3^+(y_{B_2}) + D_3 \sin n\phi\, Y_3^-(y_{B_2})\right] e^{j2\pi ft} \\ &= \rho_4(y_{B_2})\,J_n(k_{r_4} r)\left[A_4 \cos n\phi\, Y_4^+(y_{B_2}) + B_4 \sin n\phi\, Y_4^+(y_{B_2})\right] e^{j2\pi ft}\,. \end{aligned} \tag{71}$$

The time dependence is eliminated by dividing (71) through by the complex exponential term $e^{j2\pi ft}$. Carrying out the indicated multiplications and factoring reveals the following:

$$\begin{aligned} &\left[\rho_3(y_{B_2})\,J_n(k_{r_3} r)\,Y_3^+(y_{B_2})\,A_3 + \rho_3(y_{B_2})\,J_n(k_{r_3} r)\,Y_3^-(y_{B_2})\,B_3\right] \cos n\phi \\ &+ \left[\rho_3(y_{B_2})\,J_n(k_{r_3} r)\,Y_3^+(y_{B_2})\,C_3 + \rho_3(y_{B_2})\,J_n(k_{r_3} r)\,Y_3^-(y_{B_2})\,D_3\right] \sin n\phi \\ &= \left[\rho_4(y_{B_2})\,J_n(k_{r_4} r)\,Y_4^+(y_{B_2})\,A_4\right] \cos n\phi + \left[\rho_4(y_{B_2})\,J_n(k_{r_4} r)\,Y_4^+(y_{B_2})\,B_4\right] \sin n\phi\,. \end{aligned} \tag{72}$$

Setting the respective coefficients of cos $n\phi$ and sin $n\phi$ equal and rearranging yields the following pair of equations representing the fourth boundary condition (BC #4):

$$\rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, A_3 + \rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, B_3 - \rho_4(y_{B_2})\, J_n(k_{r_4} r)\, Y_4^+(y_{B_2})\, A_4 = 0\,, \tag{73}$$

and

$$\rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, C_3 + \rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, D_3 - \rho_4(y_{B_2})\, J_n(k_{r_4} r)\, Y_4^+(y_{B_2})\, B_4 = 0\,. \tag{74}$$

Again, (73) and (74) are valid only if the associated trigonometric function is not identically zero for all values of azimuthal angle ϕ.

The fifth boundary condition is that of continuity of the normal component of acoustic particle velocity across the boundary at $y = y_S$. This implies

$$U_{n1}(t,r,\phi,y_S) = U_{n2a}(t,r,\phi,y_S)\,,$$

where:

$$U_{n1}(t,r,\phi,y) = \mathbf{U}_1(t,r,\phi,y) \bullet \hat{\mathbf{n}}_s(r,\phi,y)\,, \tag{75}$$

$$U_{n2a}(t,r,\phi,y) = \mathbf{U}_{2a}(t,r,\phi,y) \bullet \hat{\mathbf{n}}_s(r,\phi,y)\,, \tag{76}$$

and $\hat{\mathbf{n}}_s(r,\phi,y)$ represents the unit vector normal to the boundary at $y = y_S$.

In order to continue with the evaluation of this boundary condition, expressions for the velocities and the unit normal vector must be developed. Recalling that the acoustic particle velocity is simply the gradient of the velocity potential implies

$$\mathbf{U}_1(t,r,\phi,y) = \nabla\ \varphi_1(t,r,\phi,y)\,, \tag{77}$$

where $\varphi_1(t,r,\phi,y)$ is given by (37), and the gradient of φ, expressed in cylindrical coordinates, is given by

$$\nabla\ \varphi(t,r,\phi,y) = \frac{\partial\varphi(t,r,\phi,y)}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\varphi(t,r,\phi,y)}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\varphi(t,r,\phi,y)}{\partial y}\hat{\mathbf{y}}\,. \tag{78}$$

Performing the indicated partial differentiations results in the following:

$$\frac{\partial\varphi_1(t,r,\phi,y)}{\partial r} = \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} Y_1^-(y)\, e^{i2\pi ft} \frac{dJ_n(k_{r_1}\ r)}{dr}$$

$$= \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} Y_1^-(y)\, e^{i2\pi ft} \frac{dJ_n(k_{r_1}\ r)}{d(k_{r_1}\ r)} \frac{d(k_{r_1}\ r)}{dr}$$

$$\frac{\partial\varphi_1(t,r,\phi,y)}{\partial r} = k_{r_1}\ \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} Y_1^-(y)\, e^{i2\pi ft} \frac{dJ_n(k_{r_1}\ r)}{d(k_{r_1}\ r)}\,, \tag{79}$$

$$\frac{\partial\varphi_1(t,r,\phi,y)}{\partial\phi} = \left\{-\,n\,B_1 \sin n\phi + n\,A_1 \cos n\phi\right\} J_n(k_{r_1} r)\, Y_1^-(y)\, e^{j2\pi ft}\,, \qquad (80)$$

and

$$\frac{\partial\varphi_1(t,r,\phi,y)}{\partial y} = \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} J_n(k_{r_1} r)\, e^{j2\pi ft}\, \frac{dY_1^-(y)}{dy}\,. \qquad (81)$$

Substituting (79) through (81) into (77) (and using (78)) yields

$$\begin{aligned} U_1(t,r,\phi,y) = \Big[& k_{r_1} \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} Y_1^-(y) \frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)}\, \hat{r} \\ & + \frac{n}{r} \left\{A_1 \cos n\phi - B_1 \sin n\phi\right\} J_n(k_{r_1} r)\, Y_1^-(y)\, \hat{\phi} \\ & + \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} J_n(k_{r_1} r) \frac{dY_1^-(y)}{dy}\, \hat{y} \Big]\, e^{j2\pi ft}\,. \end{aligned} \qquad (82)$$

Conducting a similar analysis on the appropriate expressions reveals the following set of equations for the velocities in the remaining media:

$$\begin{aligned} U_{2a}(t,r,\phi,y) = \Big[& k_{r_2} \left\{A_{2a} \cos n\phi\, Y_{2a}^+(y) + B_{2a} \cos n\phi\, Y_{2a}^-(y)\right. \\ & \left. + C_{2a} \sin n\phi\, Y_{2a}^+(y) + D_{2a} \sin n\phi\, Y_{2a}^-(y)\right\} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)}\, \hat{r} \\ & + \frac{n}{r} \left\{C_{2a} \cos n\phi\, Y_{2a}^+(y) + D_{2a} \cos n\phi\, Y_{2a}^-(y)\right. \end{aligned}$$

$$- A_{2a} \sin n\phi\, Y^{+}_{2a}(y) - B_{2a} \sin n\phi\, Y^{-}_{2a}(y)\Big\}\, J_n(k_{r_2} r)\, \hat{\boldsymbol{\phi}}$$

$$+ \Big\{ A_{2a} \cos n\phi \frac{dY^{+}_{2a}(y)}{dy} + B_{2a} \cos n\phi \frac{dY^{-}_{2a}(y)}{dy}$$

$$+ C_{2a} \sin n\phi \frac{dY^{+}_{2a}(y)}{dy} + D_{2a} \sin n\phi \frac{dY^{-}_{2a}(y)}{dy} \Big\}\, J_n(k_{r_2} r)\, \hat{\mathbf{y}} \Big]\, e^{j2\pi f t}, \qquad (83)$$

$$\mathbf{U}_{2b}(t,r,\phi,y) = \Big[k_{r_2} \Big\{ A_{2b} \cos n\phi\, Y^{+}_{2b}(y) + B_{2b} \cos n\phi\, Y^{-}_{2b}(y)$$

$$+ C_{2b} \sin n\phi\, Y^{+}_{2b}(y) + D_{2b} \sin n\phi\, Y^{-}_{2b}(y)\Big\} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)}\, \hat{\mathbf{r}}$$

$$+ \frac{n}{r} \Big\{ C_{2b} \cos n\phi\, Y^{+}_{2b}(y) + D_{2b} \cos n\phi\, Y^{-}_{2b}(y)$$

$$- A_{2b} \sin n\phi\, Y^{+}_{2b}(y) - B_{2b} \sin n\phi\, Y^{-}_{2b}(y)\Big\}\, J_n(k_{r_2} r)\, \hat{\boldsymbol{\phi}}$$

$$+ \Big\{ A_{2b} \cos n\phi \frac{dY^{+}_{2b}(y)}{dy} + B_{2b} \cos n\phi \frac{dY^{-}_{2b}(y)}{dy}$$

$$+ C_{2b} \sin n\phi \frac{dY^{+}_{2b}(y)}{dy} + D_{2b} \sin n\phi \frac{dY^{-}_{2b}(y)}{dy} \Big\}\, J_n(k_{r_2} r)\, \hat{\mathbf{y}} \Big]\, e^{j2\pi f t}, \qquad (84)$$

$$\mathbf{U}_{3}(t,r,\phi,y) = \Big[k_{r_3} \Big\{ A_{3} \cos n\phi\, Y^{+}_{3}(y) + B_{3} \cos n\phi\, Y^{-}_{3}(y)$$

$$+ C_{3} \sin n\phi\, Y^{+}_{3}(y) + D_{3} \sin n\phi\, Y^{-}_{3}(y)\Big\} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)}\, \hat{\mathbf{r}}$$

$$+\frac{n}{r}\Big\{C_3 \cos n\phi\, Y_3^+(y) + D_3 \cos n\phi\, Y_3^-(y)$$

$$- A_3 \sin n\phi\, Y_3^+(y) - B_3 \sin n\phi\, Y_3^-(y)\Big\}\, J_n(k_{r_3} r)\, \hat{\boldsymbol{\phi}}$$

$$+ \Big\{A_3 \cos n\phi \frac{dY_3^+(y)}{dy} + B_3 \cos n\phi \frac{dY_3^-(y)}{dy}$$

$$+ C_3 \sin n\phi \frac{dY_3^+(y)}{dy} + D_3 \sin n\phi \frac{dY_3^-(y)}{dy}\Big\}\, J_n(k_{r_3} r)\, \hat{\mathbf{y}}\, \Big]\, e^{j2\pi ft}, \qquad (85)$$

and

$$U_4(t,r,\phi,y) = \Big[k_{r_4} \Big\{A_4 \cos n\phi + B_4 \sin n\phi\Big\}\, Y_4^+(y) \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)}\, \hat{\mathbf{r}}$$

$$+ \frac{n}{r}\Big\{B_4 \cos n\phi - A_4 \sin n\phi\Big\}\, Y_4^+(y)\, J_n(k_{r_4} r)\, \hat{\boldsymbol{\phi}}$$

$$+ \Big\{A_4 \cos n\phi + B_4 \sin n\phi\Big\}\, J_n(k_{r_4} r) \frac{dY_4^+(y)}{dy}\, \hat{\mathbf{y}}\Big]\, e^{j2\pi ft}. \qquad (86)$$

Now, we must turn our attention to deriving an expression for the unit normal vector. A review of texts covering calculus and analytic geometry (for instance, Berkey (1988, p. 830), and Leithold (1972, p. 934)) remind us that if the surface can be expressed as a constant function of all three spatial variables (i.e., $f(r,\phi,y) =$ a constant), then the normal vector to this surface at any point is simply the gradient of the function describing the surface evaluated at that point. Recalling that both the boundaries y_S, y_{B_1}, and y_{B_2} were defined to be arbitrary functions of the other spatial variables, let

$y = f_S(r,\phi)$, (87) at the ocean surface (i.e., $y = y_S$),

$y = f_{B_1}(r,\phi)$, (88) at the ocean bottom (i.e., $y = y_{B_1}$), and

$y = f_{B_2}(r,\phi)$, (89) at the bottom layer interface (i.e., $y = y_{B_2}$).

Concentrating on the ocean surface for the moment, subtracting $f_S(r,\phi)$ from both sides of (87) yields

$$y - f_S(r,\phi) = 0 \; . \tag{90}$$

Letting a new function, $\tau_S(r,\phi,y)$, equal the left-hand side of (90), we have an equation of the form

$$\tau_S(r,\phi,y) = y - f_S(r,\phi) = 0 \; . \tag{91}$$

Taking the gradient of $\tau_S(r,\phi,y)$ using the cylindrical coordinate system gradient operator (78) yields

$$\nabla \tau_S(r,\phi,y) = \frac{\partial \tau_S(r,\phi,y)}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \tau_S(r,\phi,y)}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial \tau_S(r,\phi,y)}{\partial y}\hat{\mathbf{y}}$$

$$\nabla \tau_S(r,\phi,y) = -\frac{\partial f_S(r,\phi)}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial f_S(r,\phi)}{\partial \phi}\hat{\boldsymbol{\phi}} + \hat{\mathbf{y}} \; . \tag{92}$$

Therefore, the normal vector is given by

$$\mathbf{N}_S = -\frac{\partial f_S(r,\phi)}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial f_S(r,\phi)}{\partial \phi}\hat{\boldsymbol{\phi}} + \hat{\mathbf{y}} \; . \tag{93}$$

Since we need a unit normal vector, the magnitude of $\mathbf{N}_S$ (for which we use the symbol N_S) must be evaluated. Recalling that the magnitude of a vector is the square root of the sum of the squares of the individual components, then N_S can be written directly as

$$N_S = \sqrt{\left\{\frac{\partial f_S(r,\phi)}{\partial r}\right\}^2 + \frac{1}{r^2}\left\{\frac{\partial f_S(r,\phi)}{\partial \phi}\right\}^2 + 1}\,. \tag{94}$$

Therefore, the desired unit normal vector is given by

$$\hat{\mathbf{n}}_S = \frac{1}{N_S}\left\{-\frac{\partial f_S(r,\phi)}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial f_S(r,\phi)}{\partial \phi}\hat{\boldsymbol{\phi}} + \hat{\mathbf{y}}\right\}. \tag{95}$$

Similar analysis reveals the following expressions for the unit normal vectors at the boundaries $y = y_{B_1}$ ($\hat{\mathbf{n}}_{B_1}$) and $y = y_{B_2}$ ($\hat{\mathbf{n}}_{B_2}$):

$$\hat{\mathbf{n}}_{B_1} = \frac{1}{N_{B_1}}\left\{-\frac{\partial f_{B_1}(r,\phi)}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial f_{B_1}(r,\phi)}{\partial \phi}\hat{\boldsymbol{\phi}} + \hat{\mathbf{y}}\right\}, \tag{96}$$

where

$$N_{B_1} = \sqrt{\left\{\frac{\partial f_{B_1}(r,\phi)}{\partial r}\right\}^2 + \frac{1}{r^2}\left\{\frac{\partial f_{B_1}(r,\phi)}{\partial \phi}\right\}^2 + 1}\,, \tag{97}$$

and

$$\hat{\mathbf{n}}_{B_2} = \frac{1}{N_{B_2}}\left\{-\frac{\partial f_{B_2}(r,\phi)}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial f_{B_2}(r,\phi)}{\partial \phi}\hat{\boldsymbol{\phi}} + \hat{\mathbf{y}}\right\}, \tag{98}$$

where

$$N_{B_2} = \sqrt{\left\{\frac{\partial f_{B_2}(r,\phi)}{\partial r}\right\}^2 + \frac{1}{r^2}\left\{\frac{\partial f_{B_2}(r,\phi)}{\partial \phi}\right\}^2 + 1} \,. \tag{99}$$

Returning to the evaluation of the fifth boundary condition, substituting (82) and (95) into (75), substituting (83) and (95) into (76), and performing the indicated dot products yields

$$\begin{aligned} U_{n1}(t,r,\phi,y) = \Big[& k_{r_1} \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} Y_1^-(y) \frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)} \hat{\mathbf{r}} \\ & + \frac{n}{r}\left\{A_1 \cos n\phi - B_1 \sin n\phi\right\} J_n(k_{r_1} r)\, Y_1^-(y)\, \hat{\boldsymbol{\phi}} \\ & + \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} J_n(k_{r_1} r) \frac{dY_1^-(y)}{dy} \hat{\mathbf{y}} \Big] e^{j2\pi ft} \\ & \bullet \frac{1}{N_S}\left\{- \frac{\partial f_S(r,\phi)}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial f_S(r,\phi)}{\partial \phi}\hat{\boldsymbol{\phi}} + \hat{\mathbf{y}}\right\} \end{aligned}$$

$$\begin{aligned} U_{n1}(t,r,\phi,y) = \Big[& - k_{r_1} \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} Y_1^-(y) \frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)} \frac{\partial f_S(r,\phi)}{\partial r} \\ & - \frac{n}{r^2}\left\{A_1 \cos n\phi - B_1 \sin n\phi\right\} J_n(k_{r_1} r)\, Y_1^-(y) \frac{\partial f_S(r,\phi)}{\partial \phi} \\ & + \left\{B_1 \cos n\phi + A_1 \sin n\phi\right\} J_n(k_{r_1} r) \frac{dY_1^-(y)}{dy} \Big] \frac{e^{j2\pi ft}}{N_S} \,. \end{aligned} \tag{100}$$

$$U_{n2a}(t,r,\phi,y) = \Big[k_{r_2}\Big\{A_{2a}\cos n\phi\, Y_{2a}^{+}(y) + B_{2a}\cos n\phi\, Y_{2a}^{-}(y)$$

$$+ C_{2a}\sin n\phi\, Y_{2a}^{+}(y) + D_{2a}\sin n\phi\, Y_{2a}^{-}(y)\Big\}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)}\hat{r}$$

$$+ \frac{n}{r}\Big\{C_{2a}\cos n\phi\, Y_{2a}^{+}(y) + D_{2a}\cos n\phi\, Y_{2a}^{-}(y)$$

$$- A_{2a}\sin n\phi\, Y_{2a}^{+}(y) - B_{2a}\sin n\phi\, Y_{2a}^{-}(y)\Big\}J_n(k_{r_2}\, r)\hat{\phi}$$

$$+ \Big\{A_{2a}\cos n\phi\frac{dY_{2a}^{+}(y)}{dy} + B_{2a}\cos n\phi\frac{dY_{2a}^{-}(y)}{dy}$$

$$+ C_{2a}\sin n\phi\frac{dY_{2a}^{+}(y)}{dy} + D_{2a}\sin n\phi\frac{dY_{2a}^{-}(y)}{dy}\Big\}J_n(k_{r_2}\, r)\hat{y}\Big]e^{j2\pi ft}$$

$$\bullet\frac{1}{N_S}\Big\{-\frac{\partial f_S(r,\phi)}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial f_S(r,\phi)}{\partial\phi}\hat{\phi} + \hat{y}\Big\}$$

$$U_{n2a}(t,r,\phi,y) = \Big[-k_{r_2}\Big\{A_{2a}\cos n\phi\, Y_{2a}^{+}(y) + B_{2a}\cos n\phi\, Y_{2a}^{-}(y)$$

$$+ C_{2a}\sin n\phi\, Y_{2a}^{+}(y) + D_{2a}\sin n\phi\, Y_{2a}^{-}(y)\Big\}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)}\frac{\partial f_S(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2}\Big\{C_{2a}\cos n\phi\, Y_{2a}^{+}(y) + D_{2a}\cos n\phi\, Y_{2a}^{-}(y) - A_{2a}\sin n\phi\, Y_{2a}^{+}(y)$$

$$- B_{2a}\sin n\phi\, Y_{2a}^{-}(y)\Big\}J_n(k_{r_2}\, r)\frac{\partial f_S(r,\phi)}{\partial\phi}$$

$$+\left\{A_{2a}\cos n\phi\,\frac{dY^{+}_{2a}(y)}{dy}+B_{2a}\cos n\phi\,\frac{dY^{-}_{2a}(y)}{dy}+C_{2a}\sin n\phi\,\frac{dY^{+}_{2a}(y)}{dy}\right.$$

$$\left.+D_{2a}\sin n\phi\,\frac{dY^{-}_{2a}(y)}{dy}\right\} J_n(k_{r_2} r)\bigg]\,\frac{e^{j2\pi ft}}{N_S}\,. \tag{101}$$

Setting $U_{n1}(t,r,\phi,y_S)$ (100) equal to $U_{n2a}(t,r,\phi,y_S)$ (101), and eliminating the common term $\frac{e^{j2\pi ft}}{N_S}$ yields

$$-k_{r_1}\left\{B_1\cos n\phi+A_1\sin n\phi\right\} Y^{-}_1(y_S)\,\frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)}\,\frac{\partial f_S(r,\phi)}{\partial r}$$

$$-\frac{n}{r^2}\left\{A_1\cos n\phi-B_1\sin n\phi\right\} J_n(k_{r_1} r)\,Y^{-}_1(y_S)\,\frac{\partial f_S(r,\phi)}{\partial\phi}$$

$$+\left\{B_1\cos n\phi+A_1\sin n\phi\right\} J_n(k_{r_1} r)\,\frac{dY^{-}_1(y_S)}{dy}$$

$$=-k_{r_2}\left\{A_{2a}\cos n\phi\,Y^{+}_{2a}(y_S)+B_{2a}\cos n\phi\,Y^{-}_{2a}(y_S)\right.$$

$$\left.+C_{2a}\sin n\phi\,Y^{+}_{2a}(y_S)+D_{2a}\sin n\phi\,Y^{-}_{2a}(y_S)\right\}\frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)}\,\frac{\partial f_S(r,\phi)}{\partial r}$$

$$-\frac{n}{r^2}\left\{C_{2a}\cos n\phi\,Y^{+}_{2a}(y_S)+D_{2a}\cos n\phi\,Y^{-}_{2a}(y_S)\right.$$

$$\left.-A_{2a}\sin n\phi\,Y^{+}_{2a}(y_S)-B_{2a}\sin n\phi\,Y^{-}_{2a}(y_S)\right\} J_n(k_{r_2} r)\,\frac{\partial f_S(r,\phi)}{\partial\phi}$$

$$+\left\{A_{2a}\cos n\phi\,\frac{dY^{+}_{2a}(y_S)}{dy}+B_{2a}\cos n\phi\,\frac{dY^{-}_{2a}(y_S)}{dy}+C_{2a}\sin n\phi\,\frac{dY^{+}_{2a}(y_S)}{dy}\right.$$

$$\left.+D_{2a}\sin n\phi\,\frac{dY^{-}_{2a}(y_S)}{dy}\right\}J_n(k_{r_2}r)\,. \qquad (102)$$

Factoring (102) yields

$$\left[-k_{r_1}\frac{dJ_n(k_{r_1}r)}{d(k_{r_1}r)}Y^{-}_1(y_S)\frac{\partial f_S(r,\phi)}{\partial r}B_1-\frac{n}{r^2}J_n(k_{r_1}r)\,Y^{-}_1(y_S)\frac{\partial f_S(r,\phi)}{\partial\phi}A_1\right.$$

$$\left.+J_n(k_{r_1}r)\frac{dY^{-}_1(y_S)}{dy}B_1\right]\cos n\phi$$

$$+\left[-k_{r_1}\frac{dJ_n(k_{r_1}r)}{d(k_{r_1}r)}Y^{-}_1(y_S)\frac{\partial f_S(r,\phi)}{\partial r}A_1+\frac{n}{r^2}J_n(k_{r_1}r)\,Y^{-}_1(y_S)\frac{\partial f_S(r,\phi)}{\partial\phi}B_1\right.$$

$$\left.+J_n(k_{r_1}r)\frac{dY^{-}_1(y_S)}{dy}A_1\right]\sin n\phi$$

$$-\left[-k_{r_2}\frac{dJ_n(k_{r_2}r)}{d(k_{r_2}r)}Y^{+}_{2a}(y_S)\frac{\partial f_S(r,\phi)}{\partial r}A_{2a}-k_{r_2}\frac{dJ_n(k_{r_2}r)}{d(k_{r_2}r)}Y^{-}_{2a}(y_S)\frac{\partial f_S(r,\phi)}{\partial r}B_{2a}\right.$$

$$-\frac{n}{r^2}J_n(k_{r_2}r)\,Y^{+}_{2a}(y_S)\frac{\partial f_S(r,\phi)}{\partial\phi}C_{2a}-\frac{n}{r^2}J_n(k_{r_2}r)\,Y^{-}_{2a}(y_S)\frac{\partial f_S(r,\phi)}{\partial\phi}D_{2a}$$

$$+ J_n(k_{r_2} r) \frac{dY^+_{2a}(y_S)}{dy} A_{2a} + J_n(k_{r_2} r) \frac{dY^-_{2a}(y_S)}{dy} B_{2a} \Big] \cos n\phi$$

$$+ \Big[- k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^+_{2a}(y_S) \frac{\partial f_S(r,\phi)}{\partial r} C_{2a} - k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^-_{2a}(y_S) \frac{\partial f_S(r,\phi)}{\partial r} D_{2a}$$

$$+ \frac{n}{r^2} J_n(k_{r_2} r) Y^+_{2a}(y_S) \frac{\partial f_S(r,\phi)}{\partial \phi} A_{2a} + \frac{n}{r^2} J_n(k_{r_2} r) Y^-_{2a}(y_S) \frac{\partial f_S(r,\phi)}{\partial \phi} B_{2a}$$

$$+ J_n(k_{r_2} r) \frac{dY^+_{2a}(y_S)}{dy} C_{2a} + J_n(k_{r_2} r) \frac{dY^-_{2a}(y_S)}{dy} D_{2a} \Big] \sin n\phi \,. \qquad (103)$$

Setting the respective coefficients of cos $n\phi$ and sin $n\phi$ equal and rearranging yields the following:

$$J_n(k_{r_1} r) \frac{dY^-_1(y_S)}{dy} B_1 - k_{r_1} \frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)} Y^-_1(y_S) B_1 \frac{\partial f_S(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2} J_n(k_{r_1} r) Y^-_1(y_S) A_1 \frac{\partial f_S(r,\phi)}{\partial \phi}$$

$$= \Big[J_n(k_{r_2} r) \frac{dY^+_{2a}(y_S)}{dy} A_{2a} + J_n(k_{r_2} r) \frac{dY^-_{2a}(y_S)}{dy} B_{2a} \Big]$$

$$
- \left[k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2a}^{+}(y_S)\, A_{2a} + k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2a}^{-}(y_S)\, B_{2a}\right] \frac{\partial f_S(r,\phi)}{\partial r}
$$

$$
- \frac{n}{r^2} \left[J_n(k_{r_2} r)\, Y_{2a}^{+}(y_S)\, C_{2a} + J_n(k_{r_2} r)\, Y_{2a}^{-}(y_S)\, D_{2a}\right] \frac{\partial f_S(r,\phi)}{\partial \phi}, \qquad (104)
$$

and

$$
J_n(k_{r_1} r) \frac{dY_1^{-}(y_S)}{dy} A_1 - k_{r_1} \frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)} Y_1^{-}(y_S)\, A_1 \frac{\partial f_S(r,\phi)}{\partial r}
$$

$$
+ \frac{n}{r^2} J_n(k_{r_1} r)\, Y_1^{-}(y_S)\, B_1 \frac{\partial f_S(r,\phi)}{\partial \phi}
$$

$$
- \left[J_n(k_{r_2} r) \frac{dY_{2a}^{+}(y_S)}{dy} C_{2a} + J_n(k_{r_2} r) \frac{dY_{2a}^{-}(y_S)}{dy} D_{2a}\right]
$$

$$
- \left[k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2a}^{+}(y_S)\, C_{2a} + k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2a}^{-}(y_S)\, D_{2a}\right] \frac{\partial f_S(r,\phi)}{\partial r}
$$

$$
+ \frac{n}{r^2} \left[J_n(k_{r_2} r)\, Y_{2a}^{+}(y_S)\, A_{2a} + J_n(k_{r_2} r)\, Y_{2a}^{-}(y_S)\, B_{2a}\right] \frac{\partial f_S(r,\phi)}{\partial \phi}. \qquad (105)
$$

Again, (104) and (105) are valid only if the associated trigonometric function is not identically zero for all values of azimuthal angle ϕ. It should be noted here that both (104) and (105) are of the form

$$(\text{LHS term 1}) + (\text{LHS term 2})\frac{\partial f_S(r,\phi)}{\partial r} + (\text{LHS term 3})\frac{\partial f_S(r,\phi)}{\partial \phi}$$

$$= (\text{RHS term 1}) + (\text{RHS term 2})\frac{\partial f_S(r,\phi)}{\partial r} + (\text{RHS term 3})\frac{\partial f_S(r,\phi)}{\partial \phi}.$$

Therefore, we may simplify (104) and (105) by setting LHS term 1 equal to RHS term 1, LHS term 2 equal to RHS term 2, and LHS term 3 equal to RHS term 3. Performing this analysis and rearranging the resulting expressions yields the set of six equations representing the fifth boundary condition (BC #5)

$$J_n(k_{r_2} r)\frac{dY_{2a}^{+}(y_S)}{dy} A_{2a} + J_n(k_{r_2} r)\frac{dY_{2a}^{-}(y_S)}{dy} B_{2a} - J_n(k_{r_1} r)\frac{dY_1^{-}(y_S)}{dy} B_1 = 0\,, \tag{106}$$

$$k_{r_2}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y_{2a}^{+}(y_S)\, A_{2a} + k_{r_2}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y_{2a}^{-}(y_S)\, B_{2a}$$

$$- k_{r_1}\frac{dJ_n(k_{r_1}\, r)}{d(k_{r_1}\, r)} Y_1^{-}(y_S)\, B_1 = 0\,, \tag{107}$$

$$J_n(k_{r_2} r)\, Y_{2a}^{+}(y_S)\, C_{2a} + J_n(k_{r_2} r)\, Y_{2a}^{-}(y_S)\, D_{2a} - J_n(k_{r_1} r)\, Y_1^{-}(y_S)\, A_1 = 0\,, \tag{108}$$

$$J_n(k_{r_2} r)\frac{dY_{2a}^{+}(y_S)}{dy} C_{2a} + J_n(k_{r_2} r)\frac{dY_{2a}^{-}(y_S)}{dy} D_{2a} - J_n(k_{r_1} r)\frac{dY_1^{-}(y_S)}{dy} A_1 = 0\,, \tag{109}$$

$$k_{r_2}\frac{dJ_n(k_{r_2}\,r)}{d(k_{r_2}\,r)}\,Y_{2a}^{+}(y_S)\,C_{2a} + k_{r_2}\frac{dJ_n(k_{r_2}\,r)}{d(k_{r_2}\,r)}\,Y_{2a}^{-}(y_S)\,D_{2a}$$

$$- \, k_{r_1}\frac{dJ_n(k_{r_1}\,r)}{d(k_{r_1}\,r)}\,Y_1^{-}(y_S)\,A_1 = 0\,, \qquad (110)$$

and

$$J_n(k_{r_2}\,r)\,Y_{2a}^{+}(y_S)\,A_{2a} + J_n(k_{r_2}\,r)\,Y_{2a}^{-}(y_S)\,B_{2a} - J_n(k_{r_1}\,r)\,Y_1^{-}(y_S)\,B_1 = 0\,. \qquad (111)$$

Here, (107) and (110) are valid only if $\frac{\partial f_S(r,\phi)}{\partial r}$ is not identically zero for all values of range, r, and azimuthal angle, ϕ. Similarly, (108) and (111) are valid only if $\frac{\partial f_S(r,\phi)}{\partial \phi}$ is not identically zero for all values of range, r, and azimuthal angle, ϕ.

The sixth boundary condition is that of continuity of the normal component of the acoustic particle velocity at the boundary $y = y_{B_1}$. This implies

$$U_{n2b}(t,r,\phi,y_{B_1}) = U_{n3}(t,r,\phi,y_{B_1})\,,$$

where

$$U_{n3}(t,r,\phi,y) = \mathbf{U}_3(t,r,\phi,y) \cdot \hat{\mathbf{n}}_{B_1}(r,\phi,y)\,, \qquad (113)$$

and $\hat{\mathbf{n}}_{B_1}(r,\phi,y)$ represents the unit vector normal to the boundary at $y = y_{B_1}$.

Substituting (84) and (96) into (112), substituting (85) and (96) into (113), and performing the indicated dot products yields

$$U_{n2b}(t,r,\phi,y) = \Big[k_{r_2}\Big\{A_{2b}\cos n\phi\, Y_{2b}^{+}(y) + B_{2b}\cos n\phi\, Y_{2b}^{-}(y)$$

$$+ C_{2b}\sin n\phi\, Y_{2b}^{+}(y) + D_{2b}\sin n\phi\, Y_{2b}^{-}(y)\Big\}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)}\hat{\mathbf{r}}$$

$$+ \frac{n}{r}\Big\{C_{2b}\cos n\phi\, Y_{2b}^{+}(y) + D_{2b}\cos n\phi\, Y_{2b}^{-}(y)$$

$$- A_{2b}\sin n\phi\, Y_{2b}^{+}(y) - B_{2b}\sin n\phi\, Y_{2b}^{-}(y)\Big\}J_n(k_{r_2}\, r)\,\hat{\boldsymbol{\phi}}$$

$$+ \Big\{A_{2b}\cos n\phi\frac{dY_{2b}^{+}(y)}{dy} + B_{2b}\cos n\phi\frac{dY_{2b}^{-}(y)}{dy}$$

$$+ C_{2b}\sin n\phi\frac{dY_{2b}^{+}(y)}{dy} + D_{2b}\sin n\phi\frac{dY_{2b}^{-}(y)}{dy}\Big\}J_n(k_{r_2}\, r)\,\hat{\mathbf{y}}\Big]e^{i2\pi ft}$$

$$\bullet\frac{1}{N_{B_1}}\Big\{-\frac{\partial f_{B_1}(r,\phi)}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial f_{B_1}(r,\phi)}{\partial\phi}\hat{\boldsymbol{\phi}} + \hat{\mathbf{y}}\Big\}$$

$$U_{n2b}(t,r,\phi,y) = \Big[-k_{r_2}\Big\{A_{2b}\cos n\phi\, Y_{2b}^{+}(y) + B_{2b}\cos n\phi\, Y_{2b}^{-}(y)$$

$$+ C_{2b}\sin n\phi\, Y_{2b}^{+}(y) + D_{2b}\sin n\phi\, Y_{2b}^{-}(y)\Big\}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)}\frac{\partial f_{B_1}(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2} \Big\{ C_{2b} \cos n\phi \, Y^{+}_{2b}(y) + D_{2b} \cos n\phi \, Y^{-}_{2b}(y)$$

$$- A_{2b} \sin n\phi \, Y^{+}_{2b}(y) - B_{2b} \sin n\phi \, Y^{-}_{2b}(y) \Big\} J_n(k_{r_2} r) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi}$$

$$+ \Big\{ A_{2b} \cos n\phi \frac{dY^{+}_{2b}(y)}{dy} + B_{2b} \cos n\phi \frac{dY^{-}_{2b}(y)}{dy} + C_{2b} \sin n\phi \frac{dY^{+}_{2b}(y)}{dy}$$

$$+ D_{2b} \sin n\phi \frac{dY^{-}_{2b}(y)}{dy} \Big\} J_n(k_{r_2} r) \Big] \frac{e^{j2\pi ft}}{N_{B_1}}, \tag{114}$$

and

$$U_{n3}(t,r,\phi,y) = \Big[k_{r_3} \Big\{ A_3 \cos n\phi \, Y^{+}_3(y) + B_3 \cos n\phi \, Y^{-}_3(y)$$

$$+ C_3 \sin n\phi \, Y^{+}_3(y) + D_3 \sin n\phi \, Y^{-}_3(y) \Big\} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} \hat{r}$$

$$+ \frac{n}{r} \Big\{ C_3 \cos n\phi \, Y^{+}_3(y) + D_3 \cos n\phi \, Y^{-}_3(y)$$

$$- A_3 \sin n\phi \, Y^{+}_3(y) - B_3 \sin n\phi \, Y^{-}_3(y) \Big\} J_n(k_{r_3} r) \hat{\phi}$$

$$+ \Big\{ A_3 \cos n\phi \frac{dY^{+}_3(y)}{dy} + B_3 \cos n\phi \frac{dY^{-}_3(y)}{dy}$$

$$+ C_3 \sin n\phi \frac{dY^{+}_3(y)}{dy} + D_3 \sin n\phi \frac{dY^{-}_3(y)}{dy} \Big\} J_n(k_{r_3} r) \hat{y} \Big] e^{j2\pi ft}$$

$$\bullet \frac{1}{N_{B_1}} \left\{ - \frac{\partial f_{B_1}(r,\phi)}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} \hat{\phi} + \hat{y} \right\}$$

$$U_{n3}(t,r,\phi,y) = \Big[- k_{r_3} \Big\{ A_3 \cos n\phi \, Y_3^+(y) + B_3 \cos n\phi \, Y_3^-(y)$$

$$+ C_3 \sin n\phi \, Y_3^+(y) + D_3 \sin n\phi \, Y_3^-(y) \Big\} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} \frac{\partial f_{B_1}(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2} \Big\{ C_3 \cos n\phi \, Y_3^+(y) + D_3 \cos n\phi \, Y_3^-(y)$$

$$- A_3 \sin n\phi \, Y_3^+(y) - B_3 \sin n\phi \, Y_3^-(y) \Big\} J_n(k_{r_3} r) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi}$$

$$+ \Big\{ A_3 \cos n\phi \frac{dY_3^+(y)}{dy} + B_3 \cos n\phi \frac{dY_3^-(y)}{dy}$$

$$+ C_3 \sin n\phi \frac{dY_3^+(y)}{dy} + D_3 \sin n\phi \frac{dY_3^-(y)}{dy} \Big\} J_n(k_{r_3} r) \Big] \frac{e^{j2\pi ft}}{N_{B_1}} . \tag{115}$$

Setting $U_{n2b}(t,r,\phi,y_{B_1})$ (114) equal to $U_{n3}(t,r,\phi,y_{B_1})$ (115) and eliminating the common term $\frac{e^{j2\pi ft}}{N_{B_1}}$ yields

$$- k_{r_2} \{ A_{2b} \cos n\phi \, Y^{+}_{2b}(y_{B_1}) + B_{2b} \cos n\phi \, Y^{-}_{2b}(y_{B_1})$$

$$+ C_{2b} \sin n\phi \, Y^{+}_{2b}(y_{B_1}) + D_{2b} \sin n\phi \, Y^{-}_{2b}(y_{B_1}) \} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} \frac{\partial f_{B_1}(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2} \{ C_{2b} \cos n\phi \, Y^{+}_{2b}(y_{B_1}) + D_{2b} \cos n\phi \, Y^{-}_{2b}(y_{B_1})$$

$$- A_{2b} \sin n\phi \, Y^{+}_{2b}(y_{B_1}) - B_{2b} \sin n\phi \, Y^{-}_{2b}(y_{B_1}) \} J_n(k_{r_2} r) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi}$$

$$+ \{ A_{2b} \cos n\phi \frac{dY^{+}_{2b}(y_{B_1})}{dy} + B_{2b} \cos n\phi \frac{dY^{-}_{2b}(y_{B_1})}{dy} + C_{2b} \sin n\phi \frac{dY^{+}_{2b}(y_{B_1})}{dy}$$

$$+ D_{2b} \sin n\phi \frac{dY^{-}_{2b}(y_{B_1})}{dy} \} J_n(k_{r_2} r)$$

$$= - k_{r_3} \{ A_3 \cos n\phi \, Y^{+}_{3}(y_{B_1}) + B_3 \cos n\phi \, Y^{-}_{3}(y_{B_1})$$

$$+ C_3 \sin n\phi \, Y^{+}_{3}(y_{B_1}) + D_3 \sin n\phi \, Y^{-}_{3}(y_{B_1}) \} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} \frac{\partial f_{B_1}(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2} \{ C_3 \cos n\phi \, Y^{+}_{3}(y_{B_1}) + D_3 \cos n\phi \, Y^{-}_{3}(y_{B_1})$$

$$- A_3 \sin n\phi \, Y^{+}_{3}(y_{B_1}) - B_3 \sin n\phi \, Y^{-}_{3}(y_{B_1}) \} J_n(k_{r_3} r) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi}$$

$$+\left\{A_3 \cos n\phi \frac{dY_3^+(y_{B_1})}{dy} + B_3 \cos n\phi \frac{dY_3^-(y_{B_1})}{dy}\right.$$

$$\left. + C_3 \sin n\phi \frac{dY_3^+(y_{B_1})}{dy} + D_3 \sin n\phi \frac{dY_3^-(y_{B_1})}{dy}\right\} J_n(k_{r_3} r) . \qquad (116)$$

Factoring (116) yields

$$\left[- k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2b}^+(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial r} A_{2b} - k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2b}^-(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial r} B_{2b}\right.$$

$$- \frac{n}{r^2} J_n(k_{r_2} r)\, Y_{2b}^+(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} C_{2b} - \frac{n}{r^2} J_n(k_{r_2} r)\, Y_{2b}^-(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} D_{2b}$$

$$\left. + J_n(k_{r_2} r) \frac{dY_{2b}^+(y_{B_1})}{dy} A_{2b} + J_n(k_{r_2} r) \frac{dY_{2b}^-(y_{B_1})}{dy} B_{2b}\right] \cos n\phi$$

$$+ \left[- k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2b}^+(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial r} C_{2b} - k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2b}^-(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial r} D_{2b}\right.$$

$$+ \frac{n}{r^2} J_n(k_{r_2} r)\, Y_{2b}^+(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} A_{2b} + \frac{n}{r^2} J_n(k_{r_2} r)\, Y_{2b}^-(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} B_{2b}$$

$$+ J_n(k_{r_2} r) \frac{dY^+_{2b}(y_{B_1})}{dy} C_{2b} + J_n(k_{r_2} r) \frac{dY^-_{2b}(y_{B_1})}{dy} D_{2b} \Big] \sin n\phi$$

$$= \Big[- k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^+_3(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial r} A_3 - k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^-_3(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial r} B_3$$

$$- \frac{n}{r^2} J_n(k_{r_3} r) Y^+_3(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} C_3 - \frac{n}{r^2} J_n(k_{r_3} r) Y^-_3(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} D_3$$

$$+ J_n(k_{r_3} r) \frac{dY^+_3(y_{B_1})}{dy} A_3 + J_n(k_{r_3} r) \frac{dY^-_3(y_{B_1})}{dy} B_3 \Big] \cos n\phi$$

$$+ \Big[- k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^+_3(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial r} C_3 - k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^-_3(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial r} D_3$$

$$+ \frac{n}{r^2} J_n(k_{r_3} r) Y^+_3(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} A_3 + \frac{n}{r^2} J_n(k_{r_3} r) Y^-_3(y_{B_1}) \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} B_3$$

$$+ J_n(k_{r_3} r) \frac{dY^+_3(y_{B_1})}{dy} C_3 + J_n(k_{r_3} r) \frac{dY^-_3(y_{B_1})}{dy} D_3 \Big] \sin n\phi . \qquad (117)$$

Setting the respective coefficients of cos $n\phi$ and sin $n\phi$ equal and rearranging yields the following:

$$\left[J_n(k_{r_2} r)\frac{dY^+_{2b}(y_{B_1})}{dy}A_{2b} + J_n(k_{r_2} r)\frac{dY^-_{2b}(y_{B_1})}{dy}B_{2b}\right]$$

$$-\left[k_{r_2}\frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)}Y^+_{2b}(y_{B_1})\,A_{2b} + k_{r_2}\frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)}Y^-_{2b}(y_{B_1})\,B_{2b}\right]\frac{\partial f_{B_1}(r,\phi)}{\partial r}$$

$$-\frac{n}{r^2}\left[J_n(k_{r_2} r)\,Y^+_{2b}(y_{B_1})\,C_{2b} + J_n(k_{r_2} r)\,Y^-_{2b}(y_{B_1})\,D_{2b}\right]\frac{\partial f_{B_1}(r,\phi)}{\partial \phi}$$

$$= \left[J_n(k_{r_3} r)\frac{dY^+_3(y_{B_1})}{dy}A_3 + J_n(k_{r_3} r)\frac{dY^-_3(y_{B_1})}{dy}B_3\right]$$

$$-\left[k_{r_3}\frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)}Y^+_3(y_{B_1})\,A_3 + k_{r_3}\frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)}Y^-_3(y_{B_1})\,B_3\right]\frac{\partial f_{B_1}(r,\phi)}{\partial r}$$

$$-\frac{n}{r^2}\left[J_n(k_{r_3} r)\,Y^+_3(y_{B_1})\,C_3 + J_n(k_{r_3} r)\,Y^-_3(y_{B_1})\,D_3\right]\frac{\partial f_{B_1}(r,\phi)}{\partial \phi}. \qquad (118)$$

and

$$\left[J_n(k_{r_2} r) \frac{dY^+_{2b}(y_{B_1})}{dy} C_{2b} + J_n(k_{r_2} r) \frac{dY^-_{2b}(y_{B_1})}{dy} D_{2b}\right]$$

$$- \left[k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^+_{2b}(y_{B_1}) C_{2b} + k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^-_{2b}(y_{B_1}) D_{2b}\right] \frac{\partial f_{B_1}(r,\phi)}{\partial r}$$

$$+ \frac{n}{r^2} \left[J_n(k_{r_2} r) Y^+_{2b}(y_{B_1}) A_{2b} + J_n(k_{r_2} r) Y^-_{2b}(y_{B_1}) B_{2b}\right] \frac{\partial f_{B_1}(r,\phi)}{\partial \phi}$$

$$- \left[J_n(k_{r_3} r) \frac{dY^+_3(y_{B_1})}{dy} C_3 + J_n(k_{r_3} r) \frac{dY^-_3(y_{B_1})}{dy} D_3\right]$$

$$- \left[k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^+_3(y_{B_1}) C_3 + k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^-_3(y_{B_1}) D_3\right] \frac{\partial f_{B_1}(r,\phi)}{\partial r}$$

$$+ \frac{n}{r^2} \left[J_n(k_{r_3} r) Y^+_3(y_{B_1}) A_3 + J_n(k_{r_3} r) Y^-_3(y_{B_1}) B_3\right] \frac{\partial f_{B_1}(r,\phi)}{\partial \phi} . \qquad (119)$$

Again, (118) and (119) are valid only if the associated trigonometric function is not identically zero for all values of azimuthal angle ϕ. Conducting an analysis of (118) and (119) similar to that of the previous boundary condition yields the set of six equations representing the sixth boundary condition (BC # 6)

$$J_n(k_{r_2} r)\frac{dY^{+}_{2b}(y_{B_1})}{dy} A_{2b} + J_n(k_{r_2} r)\frac{dY^{-}_{2b}(y_{B_1})}{dy} B_{2b}$$

$$- J_n(k_{r_3} r)\frac{dY^{+}_{3}(y_{B_1})}{dy} A_{3} - J_n(k_{r_3} r)\frac{dY^{-}_{3}(y_{B_1})}{dy} B_{3} = 0\,, \tag{120}$$

$$k_{r_2}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y^{+}_{2b}(y_{B_1})\, A_{2b} + k_{r_2}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y^{-}_{2b}(y_{B_1})\, B_{2b}$$

$$- k_{r_3}\frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)} Y^{+}_{3}(y_{B_1})\, A_{3} - k_{r_3}\frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)} Y^{-}_{3}(y_{B_1})\, B_{3} - 0\,, \tag{121}$$

$$J_n(k_{r_2} r)\, Y^{+}_{2b}(y_{B_1})\, C_{2b} + J_n(k_{r_2} r)\, Y^{-}_{2b}(y_{B_1})\, D_{2b}$$

$$- J_n(k_{r_3} r)\, Y^{+}_{3}(y_{B_1})\, C_{3} - J_n(k_{r_3} r)\, Y^{-}_{3}(y_{B_1})\, D_{3} = 0\,, \tag{122}$$

$$J_n(k_{r_2} r) \frac{dY^{+}_{2b}(y_{B_1})}{dy} C_{2b} + J_n(k_{r_2} r) \frac{dY^{-}_{2b}(y_{B_1})}{dy} D_{2b}$$

$$- J_n(k_{r_3} r) \frac{dY^{+}_{3}(y_{B_1})}{dy} C_3 - J_n(k_{r_3} r) \frac{dY^{-}_{3}(y_{B_1})}{dy} D_3 = 0 , \qquad (123)$$

$$k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^{+}_{2b}(y_{B_1}) C_{2b} + k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^{-}_{2b}(y_{B_1}) D_{2b}$$

$$- k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^{+}_{3}(y_{B_1}) C_3 - k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^{-}_{3}(y_{B_1}) D_3 = 0 , \qquad (124)$$

and

$$J_n(k_{r_2} r)\, Y^{+}_{2b}(y_{B_1})\, A_{2b} + J_n(k_{r_2} r)\, Y^{-}_{2b}(y_{B_1})\, B_{2b}$$

$$- J_n(k_{r_3} r)\, Y^{+}_{3}(y_{B_1})\, A_3 - J_n(k_{r_3} r)\, Y^{-}_{3}(y_{B_1})\, B_3 = 0 . \qquad (125)$$

Here, (121) and (124) are valid only if $\frac{\partial f_{B_1}(r,\phi)}{\partial r}$ is not identically zero for all values of range, r, and azimuthal angle, ϕ. Similarly, (122) and (125)

are valid only if $\dfrac{\partial f_{B_1}(r,\phi)}{\partial \phi}$ is not identically zero for all values of range, r, and azimuthal angle, ϕ.

The seventh boundary condition is that of continuity of the normal component of the acoustic particle velocity at the boundary $y = y_{B_2}$. This implies

$$U_{n3}(t,r,\phi,y_{B_2}) = U_{n4}(t,r,\phi,y_{B_2}) \,,$$

where

$$U_{n3}(t,r,\phi,y) = \mathbf{U}_3(t,r,\phi,y) \cdot \hat{\mathbf{n}}_{B_2}(r,\phi,y) \,, \tag{126}$$

$$U_{n4}(t,r,\phi,y) = \mathbf{U}_4(t,r,\phi,y) \cdot \hat{\mathbf{n}}_{B_2}(r,\phi,y) \,, \tag{127}$$

and $\hat{\mathbf{n}}_{B_2}(r,\phi,y)$ represents the unit vector normal to the boundary at $y = y_{B_2}$.

Substituting (85) and (98) into (126), substituting (86) and (98) into (127), and performing the indicated dot products yields

$$U_{n3}(t,r,\phi,y) = \Big[k_{r_3}\Big\{A_3\cos n\phi\, Y_3^+(y) + B_3\cos n\phi\, Y_3^-(y)$$

$$+ C_3\sin n\phi\, Y_3^+(y) + D_3\sin n\phi\, Y_3^-(y)\Big\}\frac{dJ_n(k_{r_3}r)}{d(k_{r_3}r)}\hat{r}$$

$$+\frac{n}{r}\Big\{C_3\cos n\phi\, Y_3^+(y) + D_3\cos n\phi\, Y_3^-(y)$$

$$- A_3\sin n\phi\, Y_3^+(y) - B_3\sin n\phi\, Y_3^-(y)\Big\} J_n(k_{r_3}r)\hat{\phi}$$

$$+\Big\{A_3\cos n\phi\frac{dY_3^+(y)}{dy} + B_3\cos n\phi\frac{dY_3^-(y)}{dy}$$

$$+ C_3\sin n\phi\frac{dY_3^+(y)}{dy} + D_3\sin n\phi\frac{dY_3^-(y)}{dy}\Big\} J_n(k_{r_3}r)\hat{y}\Big]e^{j2\pi ft}$$

$$\bullet\frac{1}{N_{B_2}}\Big\{-\frac{\partial f_{B_2}(r,\phi)}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial f_{B_2}(r,\phi)}{\partial\phi}\hat{\phi} + \hat{y}\Big\}$$

$$U_{n3}(t,r,\phi,y) = \Big[-k_{r_3}\Big\{A_3\cos n\phi\, Y_3^+(y) + B_3\cos n\phi\, Y_3^-(y)$$

$$+ C_3\sin n\phi\, Y_3^+(y) + D_3\sin n\phi\, Y_3^-(y)\Big\}\frac{dJ_n(k_{r_3}r)}{d(k_{r_3}r)}\frac{\partial f_{B_2}(r,\phi)}{\partial r}$$

$$-\frac{n}{r^2}\Big\{C_3\cos n\phi\, Y_3^+(y) + D_3\cos n\phi\, Y_3^-(y)$$

$$- A_3\sin n\phi\, Y_3^+(y) - B_3\sin n\phi\, Y_3^-(y)\Big\} J_n(k_{r_3}r)\frac{\partial f_{B_2}(r,\phi)}{\partial\phi}$$

$$+\left\{A_3 \cos n\phi \frac{dY_3^+(y)}{dy} + B_3 \cos n\phi \frac{dY_3^-(y)}{dy}\right.$$

$$\left.+ C_3 \sin n\phi \frac{dY_3^+(y)}{dy} + D_3 \sin n\phi \frac{dY_3^-(y)}{dy}\right\} J_n(k_{r_3} r)\Big] \frac{e^{i2\pi ft}}{N_{B_2}}, \qquad (128)$$

and

$$U_{n4}(t,r,\phi,y) = \Big[k_{r_4} \left\{A_4 \cos n\phi + B_4 \sin n\phi\right\} Y_4^+(y) \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} \hat{r}$$

$$+ \frac{n}{r} \left\{B_4 \cos n\phi - A_4 \sin n\phi\right\} Y_4^+(y) J_n(k_{r_4} r) \hat{\phi}$$

$$+ \left\{A_4 \cos n\phi + B_4 \sin n\phi\right\} J_n(k_{r_4} r) \frac{dY_4^+(y)}{dy} \hat{y}\Big] e^{i2\pi ft}$$

$$\bullet \frac{1}{N_{B_2}} \left\{- \frac{\partial f_{B_2}(r,\phi)}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} \hat{\phi} + \hat{y}\right\}$$

$$U_{n4}(t,r,\phi,y) = \Big[- k_{r_4} \left\{A_4 \cos n\phi + B_4 \sin n\phi\right\} Y_4^+(y) \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} \frac{\partial f_{B_2}(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2} \left\{B_4 \cos n\phi - A_4 \sin n\phi\right\} Y_4^+(y) J_n(k_{r_4} r) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi}$$

$$+ \left\{A_4 \cos n\phi + B_4 \sin n\phi\right\} J_n(k_{r_4} r) \frac{dY_4^+(y)}{dy}\Big] \frac{e^{i2\pi ft}}{N_{B_2}}. \qquad (129)$$

Setting $U_{n3}(t,r,\phi,y_{B_2})$ (128) equal to $U_{n4}(t,r,\phi,y_{B_2})$ (129) and eliminating the common term $\dfrac{e^{j2\pi ft}}{N_{B_2}}$ reveals

$$
\begin{aligned}
&- k_{r_3} \Big\{ A_3 \cos n\phi\, Y_3^+(y_{B_2}) + B_3 \cos n\phi\, Y_3^-(y_{B_2}) \\
&+ C_3 \sin n\phi\, Y_3^+(y_{B_2}) + D_3 \sin n\phi\, Y_3^-(y_{B_2}) \Big\} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} \frac{\partial f_{B_2}(r,\phi)}{\partial r} \\
&- \frac{n}{r^2} \Big\{ C_3 \cos n\phi\, Y_3^+(y_{B_2}) + D_3 \cos n\phi\, Y_3^-(y_{B_2}) \\
&- A_3 \sin n\phi\, Y_3^+(y_{B_2}) - B_3 \sin n\phi\, Y_3^-(y_{B_2}) \Big\} J_n(k_{r_3} r) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} \\
&+ \Big\{ A_3 \cos n\phi \frac{dY_3^+(y_{B_2})}{dy} + B_3 \cos n\phi \frac{dY_3^-(y_{B_2})}{dy} \\
&+ C_3 \sin n\phi \frac{dY_3^+(y_{B_2})}{dy} + D_3 \sin n\phi \frac{dY_3^-(y_{B_2})}{dy} \Big\} J_n(k_{r_3} r)
\end{aligned}
$$

$$
\begin{aligned}
&= - k_{r_4} \Big\{ A_4 \cos n\phi + B_4 \sin n\phi \Big\} Y_4^+(y_{B_2}) \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} \frac{\partial f_{B_2}(r,\phi)}{\partial r} \\
&- \frac{n}{r^2} \Big\{ B_4 \cos n\phi - A_4 \sin n\phi \Big\} Y_4^+(y_{B_2}) J_n(k_{r_4} r) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi}
\end{aligned}
$$

$$+ \left\{ A_4 \cos n\phi + B_4 \sin n\phi \right\} J_n(k_{r_4} r) \frac{dY_4^+(y_{B_2})}{dy} . \qquad (130)$$

Factoring (130) yields

$$\Big[- k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^+(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial r} A_3 - k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^-(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial r} B_3$$

$$- \frac{n}{r^2} J_n(k_{r_3} r) Y_3^+(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} C_3 - \frac{n}{r^2} J_n(k_{r_3} r) Y_3^-(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} D_3$$

$$+ J_n(k_{r_3} r) \frac{dY_3^+(y_{B_2})}{dy} A_3 + J_n(k_{r_3} r) \frac{dY_3^-(y_{B_2})}{dy} B_3 \Big] \cos n\phi$$

$$+ \Big[- k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^+(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial r} C_3 - k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^-(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial r} D_3$$

$$+ \frac{n}{r^2} J_n(k_{r_3} r) Y_3^+(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} A_3 + \frac{n}{r^2} J_n(k_{r_3} r) Y_3^-(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} B_3$$

$$+ J_n(k_{r_3} r) \frac{dY_3^+(y_{B_2})}{dy} C_3 + J_n(k_{r_3} r) \frac{dY_3^-(y_{B_2})}{dy} D_3 \Big] \sin n\phi$$

$$= \Big[- k_{r_4} \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} Y_4^+(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial r} A_4 - \frac{n}{r^2} J_n(k_{r_4} r) Y_4^+(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} B_4$$

$$+ J_n(k_{r_4} r) \frac{dY_4^+(y_{B_2})}{dy} A_4 \Big] \cos n\phi + \Big[- k_{r_4} \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} Y_4^+(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial r} B_4$$

$$+ \frac{n}{r^2} J_n(k_{r_4} r) Y_4^+(y_{B_2}) \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} A_4 + J_n(k_{r_4} r) \frac{dY_4^+(y_{B_2})}{dy} B_4 \Big] \sin n\phi . \quad (131)$$

Setting the respective coefficients of cos $n\phi$ and sin $n\phi$ equal and rearranging yields the following:

$$\Big[J_n(k_{r_3} r) \frac{dY_3^+(y_{B_2})}{dy} A_3 + J_n(k_{r_3} r) \frac{dY_3^-(y_{B_2})}{dy} B_3 \Big]$$

$$- \Big[k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^+(y_{B_2}) A_3 + k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^-(y_{B_2}) B_3 \Big] \frac{\partial f_{B_2}(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2} \left[J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, C_3 + J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, D_3 \right] \frac{\partial f_{B_2}(r,\phi)}{\partial \phi}$$

$$- J_n(k_{r_4} r) \frac{dY_4^+(y_{B_2})}{dy} A_4 - k_{r_4} \frac{dJ_n(k_{r_4}\, r)}{d(k_{r_4}\, r)} Y_4^+(y_{B_2})\, A_4 \frac{\partial f_{B_2}(r,\phi)}{\partial r}$$

$$- \frac{n}{r^2} J_n(k_{r_4} r)\, Y_4^+(y_{B_2})\, B_4 \frac{\partial f_{B_2}(r,\phi)}{\partial \phi}\,. \qquad (132)$$

and

$$\left[J_n(k_{r_3} r) \frac{dY_3^+(y_{B_2})}{dy} C_3 + J_n(k_{r_3} r) \frac{dY_3^-(y_{B_2})}{dy} D_3 \right]$$

$$- \left[k_{r_3} \frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)} Y_3^+(y_{B_2})\, C_3 + k_{r_3} \frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)} Y_3^-(y_{B_2})\, D_3 \right] \frac{\partial f_{B_2}(r,\phi)}{\partial r}$$

$$+ \frac{n}{r^2} \left[J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, A_3 + J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, B_3 \right] \frac{\partial f_{B_2}(r,\phi)}{\partial \phi}$$

$$
- J_n(k_{r_4} r) \frac{dY_4^+(y_{B_2})}{dy} B_4 - k_{r_4} \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} Y_4^+(y_{B_2}) B_4 \frac{\partial f_{B_2}(r,\phi)}{\partial r}
$$

$$
+ \frac{n}{r^2} J_n(k_{r_4} r) Y_4^+(y_{B_2}) A_4 \frac{\partial f_{B_2}(r,\phi)}{\partial \phi} . \qquad (133)
$$

Again, (132) and (133) are valid only if the associated trigonometric function is not identically zero for all values of azimuthal angle ϕ. Conducting an analysis of (132) and (133) similar to that of the previous two boundary conditions yields the set of six equations representing the seventh boundary condition (BC # 7)

$$
J_n(k_{r_3} r) \frac{dY_3^+(y_{B_2})}{dy} A_3 + J_n(k_{r_3} r) \frac{dY_3^-(y_{B_2})}{dy} B_3 - J_n(k_{r_4} r) \frac{dY_4^+(y_{B_2})}{dy} A_4 = 0 \ , \qquad (134)
$$

$$
k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^+(y_{B_2}) A_3 + k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^-(y_{B_2}) B_3
$$

$$
- k_{r_4} \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} Y_4^+(y_{B_2}) A_4 = 0 \ . \qquad (135)
$$

$$J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, C_3 + J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, D_3 - J_n(k_{r_4} r)\, Y_4^+(y_{B_2})\, B_4 = 0\,, \quad (136)$$

$$J_n(k_{r_3} r) \frac{dY_3^+(y_{B_2})}{dy} C_3 + J_n(k_{r_3} r) \frac{dY_3^-(y_{B_2})}{dy} D_3 - J_n(k_{r_4} r) \frac{dY_4^+(y_{B_2})}{dy} B_4 = 0\,, \quad (137)$$

$$k_{r_3} \frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)} Y_3^+(y_{B_2})\, C_3 + k_{r_3} \frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)} Y_3^-(y_{B_2})\, D_3$$

$$- k_{r_4} \frac{dJ_n(k_{r_4}\, r)}{d(k_{r_4}\, r)} Y_4^+(y_{B_2})\, B_4 = 0\,, \quad (138)$$

and

$$J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, A_3 + J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, B_3 - J_n(k_{r_4} r)\, Y_4^+(y_{B_2})\, A_4 = 0\,. \quad (139)$$

Here, (135) and (138) are valid only if $\frac{\partial f_{B_2}(r,\phi)}{\partial r}$ is not identically zero for all values of range, r, and azimuthal angle, ϕ. Similarly, (136) and (139) are valid only if $\frac{\partial f_{B_2}(r,\phi)}{\partial \phi}$ is not identically zero for all values of range, r, and azimuthal angle, ϕ.

The final boundary condition is that of discontinuity of the normal component of the acoustic particle velocity across the boundary at $y = y_0$. This implies

$$U'_{n2a}(t,r,\phi,y_0) = U'_{n2b}(t,r,\phi,y_0) + \left\{G_1 \cos n\phi + G_2 \sin n\phi\right\} J_n(k_{r_2} r)\, e^{j2\pi ft}, \quad (140)$$

where:

$$U'_{n2a}(t,r,\phi,y) = \mathbf{U}_{2a}(t,r,\phi,y) \bullet \hat{\mathbf{n}}_0(r,\phi,y), \quad (141)$$

$$U'_{n2b}(t,r,\phi,y) = \mathbf{U}_{2b}(t,r,\phi,y) \bullet \hat{\mathbf{n}}_0(r,\phi,y), \quad (142)$$

$$\hat{\mathbf{n}}_0(r,\phi,y) = \hat{\mathbf{y}}, \quad (143)$$

(143) represents the unit normal vector to the boundary at $y = y_0$, and G_1 and G_2 represent amounts of discontinuity. The "prime" superscripts (in U'_{n2a} and U'_{n2b}) are used to indicate that these velocities are to be evaluated at $y = y_0$ (This was necessary since the notation U_{n2a} was used in the evaluation of the velocity boundary condition at the surface, and the notation U_{n2b} was used in the evaluation of the velocity boundary condition at the boundary $y = y_{B_1}$).

Substituting (83) and (143) into (141), substituting (84) and (143) into (142), and performing the indicated dot products yields

$$U'_{n2a}(t,r,\phi,y) = \Big[k_{r_2} \Big\{ A_{2a} \cos n\phi \, Y^+_{2a}(y) + B_{2a} \cos n\phi \, Y^-_{2a}(y) + C_{2a} \sin n\phi \, Y^+_{2a}(y)$$

$$+ D_{2a} \sin n\phi \, Y^-_{2a}(y) \Big\} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} \hat{\mathbf{r}} + \frac{n}{r} \Big\{ C_{2a} \cos n\phi \, Y^+_{2a}(y)$$

$$+ D_{2a} \cos n\phi \, Y^-_{2a}(y) - A_{2a} \sin n\phi \, Y^+_{2a}(y) - B_{2a} \sin n\phi \, Y^-_{2a}(y) \Big\} J_n(k_{r_2} r) \hat{\boldsymbol{\phi}}$$

$$+ \Big\{ A_{2a} \cos n\phi \frac{dY^+_{2a}(y)}{dy} + B_{2a} \cos n\phi \frac{dY^-_{2a}(y)}{dy}$$

$$+ C_{2a} \sin n\phi \frac{dY^+_{2a}(y)}{dy} + D_{2a} \sin n\phi \frac{dY^-_{2a}(y)}{dy} \Big\} J_n(k_{r_2} r) \hat{\mathbf{y}} \Big] e^{j2\pi ft} \bullet \hat{\mathbf{y}}$$

$$U'_{n2a}(t,r,\phi,y) = \Big\{ A_{2a} \cos n\phi \frac{dY^+_{2a}(y)}{dy} + B_{2a} \cos n\phi \frac{dY^-_{2a}(y)}{dy}$$

$$+ C_{2a} \sin n\phi \frac{dY^+_{2a}(y)}{dy} + D_{2a} \sin n\phi \frac{dY^-_{2a}(y)}{dy} \Big\} J_n(k_{r_2} r) e^{j2\pi ft} , \qquad (144)$$

and

$$U'_{n2b}(t,r,\phi,y) = \Big[k_{r_2} \Big\{ A_{2b} \cos n\phi \, Y^+_{2b}(y) + B_{2b} \cos n\phi \, Y^-_{2b}(y)$$

$$+ C_{2b} \sin n\phi \, Y^+_{2b}(y) + D_{2b} \sin n\phi \, Y^-_{2b}(y) \Big\} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} \hat{\mathbf{r}}$$

$$+ \frac{n}{r} \Big\{ C_{2b} \cos n\phi \, Y^+_{2b}(y) + D_{2b} \cos n\phi \, Y^-_{2b}(y)$$

$$- A_{2b} \sin n\phi \, Y_{2b}^{+}(y) - B_{2b} \sin n\phi \, Y_{2b}^{-}(y) \Big\} J_n(k_{r_2} r) \, \hat{\phi}$$

$$+ \Big\{ A_{2b} \cos n\phi \frac{dY_{2b}^{+}(y)}{dy} + B_{2b} \cos n\phi \frac{dY_{2b}^{-}(y)}{dy}$$

$$+ C_{2b} \sin n\phi \frac{dY_{2b}^{+}(y)}{dy} + D_{2b} \sin n\phi \frac{dY_{2b}^{-}(y)}{dy} \Big\} J_n(k_{r_2} r) \, \hat{y} \Big] e^{j2\pi ft} \cdot \hat{y}$$

$$U'_{n2b}(t,r,\phi,y) = \Big\{ A_{2b} \cos n\phi \frac{dY_{2b}^{+}(y)}{dy} + B_{2b} \cos n\phi \frac{dY_{2b}^{-}(y)}{dy}$$

$$+ C_{2b} \sin n\phi \frac{dY_{2b}^{+}(y)}{dy} + D_{2b} \sin n\phi \frac{dY_{2b}^{-}(y)}{dy} \Big\} J_n(k_{r_2} r) \, e^{j2\pi ft} . \qquad (145)$$

Substituting (144) and (145) into (140), dividing out the common terms, and evaluating the resulting expressions at $y = y_0$ yields

$$\Big\{ \frac{dY_{2a}^{+}(y_0)}{dy} A_{2a} + \frac{dY_{2a}^{-}(y_0)}{dy} B_{2a} - \frac{dY_{2b}^{+}(y_0)}{dy} A_{2b} - \frac{dY_{2b}^{-}(y_0)}{dy} B_{2b} \Big\} \cos n\phi$$

$$+ \Big\{ \frac{dY_{2a}^{+}(y_0)}{dy} C_{2a} + \frac{dY_{2a}^{-}(y_0)}{dy} D_{2a} - \frac{dY_{2b}^{+}(y_0)}{dy} C_{2b} - \frac{dY_{2b}^{-}(y_0)}{dy} D_{2b} \Big\} \sin n\phi$$

$$= G_1 \cos n\phi + G_2 \sin n\phi . \qquad (146)$$

Setting the respective coefficients of cos $n\phi$ and sin $n\phi$ equal and rearranging yields the following pair of equations representing the eighth and final boundary condition (BC #8):

$$\left\{\frac{dY_{2a}^{+}(y_0)}{dy} A_{2a} + \frac{dY_{2a}^{-}(y_0)}{dy} B_{2a} - \frac{dY_{2b}^{+}(y_0)}{dy} A_{2b} - \frac{dY_{2b}^{-}(y_0)}{dy} B_{2b}\right\} = G_1 , \qquad (147)$$

and

$$\left\{\frac{dY_{2a}^{+}(y_0)}{dy} C_{2a} + \frac{dY_{2a}^{-}(y_0)}{dy} D_{2a} - \frac{dY_{2b}^{+}(y_0)}{dy} C_{2b} - \frac{dY_{2b}^{-}(y_0)}{dy} D_{2b}\right\} = G_2 . \qquad (148)$$

Again, (147) and (148) are valid only if the associated trigonometric function is not identically zero for all values of azimuthal angle ϕ.

F. SUMMARY OF BOUNDARY CONDITION EQUATIONS AND THEIR VALIDITY

To summarize, the boundary condition equations which must be satisfied for our general waveguide model are as follows:

$$\rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^{+}(y_S)\, A_{2a} + \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^{-}(y_S)\, B_{2a}$$

$$- \rho_1(y_S)\, J_n(k_{r_1} r)\, Y_1^{-}(y_S)\, B_1 = 0 , \qquad (149)$$

which is valid only if cos $n\phi$ is not identically zero for all values of ϕ.

$$\rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^{+}(y_S)\, C_{2a} + \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^{-}(y_S)\, D_{2a}$$

$$- \rho_1(y_S)\, J_n(k_{r_1} r)\, Y_1^{-}(y_S)\, A_1 = 0 , \qquad (150)$$

which is valid only if sin $n\phi$ is not identically zero for all values of ϕ.

$$Y_{2a}^{+}(y_o)\, A_{2a} + Y_{2a}^{-}(y_o)\, B_{2a} - Y_{2b}^{+}(y_o)\, A_{2b} - Y_{2b}^{-}(y_o)\, B_{2b} = 0\,, \tag{151}$$

which is valid only if cos nϕ is not identically zero for all values of ϕ.

$$Y_{2a}^{+}(y_o)\, C_{2a} + Y_{2a}^{-}(y_o)\, D_{2a} - Y_{2b}^{+}(y_o)\, C_{2b} - Y_{2b}^{-}(y_o)\, D_{2b} = 0\,, \tag{152}$$

which is valid only if sin nϕ is not identically zero for all values of ϕ.

$$\begin{aligned} &\rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^{+}(y_{B_1})\, A_{2b} + \rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^{-}(y_{B_1})\, B_{2b} \\ &- \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^{+}(y_{B_1})\, A_3 - \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^{-}(y_{B_1})\, B_3 = 0\,, \end{aligned} \tag{153}$$

which is valid only if cos nϕ is not identically zero for all values of ϕ.

$$\begin{aligned} &\rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^{+}(y_{B_1})\, C_{2b} + \rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^{-}(y_{B_1})\, D_{2b} \\ &- \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^{+}(y_{B_1})\, C_3 - \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^{-}(y_{B_1})\, D_3 = 0\,, \end{aligned} \tag{154}$$

which is valid only if sin nϕ is not identically zero for all values of ϕ.

$$\begin{aligned} &\rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^{+}(y_{B_2})\, A_3 + \rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^{-}(y_{B_2})\, B_3 \\ &\qquad - \rho_4(y_{B_2})\, J_n(k_{r_4} r)\, Y_4^{+}(y_{B_2})\, A_4 = 0\,, \end{aligned} \tag{155}$$

which is valid only if cos nϕ is not identically zero for all values of ϕ.

$$\rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, C_3 + \rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, D_3$$

$$- \rho_4(y_{B_2})\, J_n(k_{r_4} r)\, Y_4^+(y_{B_2})\, B_4 = 0\,, \qquad (156)$$

which is valid only if sin nϕ is not identically zero for all values of ϕ.

$$J_n(k_{r_2} r)\frac{dY_{2a}^+(y_S)}{dy} A_{2a} + J_n(k_{r_2} r)\frac{dY_{2a}^-(y_S)}{dy} B_{2a} - J_n(k_{r_1} r)\frac{dY_1^-(y_S)}{dy} B_1 = 0\,, \quad (157)$$

which is valid only if cos nϕ is not identically zero for all values of ϕ.

$$J_n(k_{r_2} r)\frac{dY_{2a}^+(y_S)}{dy} C_{2a} + J_n(k_{r_2} r)\frac{dY_{2a}^-(y_S)}{dy} D_{2a} - J_n(k_{r_1} r)\frac{dY_1^-(y_S)}{dy} A_1 = 0\,, \quad (158)$$

which is valid only if sin nϕ is not identically zero for all values of ϕ.

$$k_{r_2}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y_{2a}^+(y_S)\, A_{2a} + k_{r_2}\frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y_{2a}^-(y_S)\, B_{2a}$$

$$- k_{r_1}\frac{dJ_n(k_{r_1}\, r)}{d(k_{r_1}\, r)} Y_1^-(y_S)\, B_1 = 0\,, \qquad (159)$$

which is valid only if cos nϕ and $\frac{\partial f_S(r,\phi)}{\partial r}$ are not identically zero for all values of r and ϕ.

$$J_n(k_{r_2} r)\, Y^{+}_{2a}(y_S)\, C_{2a} + J_n(k_{r_2} r)\, Y^{-}_{2a}(y_S)\, D_{2a} - J_n(k_{r_1} r)\, Y^{-}_{1}(y_S)\, A_1 = 0\,, \qquad (160)$$

which is valid only if cos $n\phi$ and $\dfrac{\partial f_S(r,\phi)}{\partial \phi}$ are not identically zero for all values of r and ϕ.

$$k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^{+}_{2a}(y_S)\, C_{2a} + k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^{-}_{2a}(y_S)\, D_{2a}$$

$$- k_{r_1} \frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)} Y^{-}_{1}(y_S)\, A_1 = 0\,, \qquad (161)$$

which is valid only if sin $n\phi$ and $\dfrac{\partial f_S(r,\phi)}{\partial r}$ are not identically zero for all values of r and ϕ.

$$J_n(k_{r_2} r)\, Y^{+}_{2a}(y_S)\, A_{2a} + J_n(k_{r_2} r)\, Y^{-}_{2a}(y_S)\, B_{2a} - J_n(k_{r_1} r)\, Y^{-}_{1}(y_S)\, B_1 = 0\,, \qquad (162)$$

which is valid only if sin $n\phi$ and $\dfrac{\partial f_S(r,\phi)}{\partial \phi}$ are not identically zero for all values of r and ϕ.

$$J_n(k_{r_2} r) \frac{dY^+_{2b}(y_{B_1})}{dy} A_{2b} + J_n(k_{r_2} r) \frac{dY^-_{2b}(y_{B_1})}{dy} B_{2b}$$

$$- J_n(k_{r_3} r) \frac{dY^+_3(y_{B_1})}{dy} A_3 - J_n(k_{r_3} r) \frac{dY^-_3(y_{B_1})}{dy} B_3 = 0 \,, \qquad (163)$$

which is valid only if cos $n\phi$ is not identically zero for all values of ϕ.

$$J_n(k_{r_2} r) \frac{dY^+_{2b}(y_{B_1})}{dy} C_{2b} + J_n(k_{r_2} r) \frac{dY^-_{2b}(y_{B_1})}{dy} D_{2b}$$

$$- J_n(k_{r_3} r) \frac{dY^+_3(y_{B_1})}{dy} C_3 - J_n(k_{r_3} r) \frac{dY^-_3(y_{B_1})}{dy} D_3 = 0 \,, \qquad (164)$$

which is valid only if sin $n\phi$ is not identically zero for all values of ϕ.

$$k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^+_{2b}(y_{B_1}) A_{2b} + k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y^-_{2b}(y_{B_1}) B_{2b}$$

$$- k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^+_3(y_{B_1}) A_3 - k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y^-_3(y_{B_1}) B_3 = 0 \,, \qquad (165)$$

which is valid only if cos $n\phi$ and $\dfrac{\partial f_{B_1}(r,\phi)}{\partial r}$ are not identically zero for all values of r and ϕ.

$$J_n(k_{r_2} r)\, Y^+_{2b}(y_{B_1})\, C_{2b} + J_n(k_{r_2} r)\, Y^-_{2b}(y_{B_1})\, D_{2b}$$

$$- J_n(k_{r_3} r)\, Y^+_3(y_{B_1})\, C_3 - J_n(k_{r_3} r)\, Y^-_3(y_{B_1})\, D_3 = 0\,, \qquad (166)$$

which is valid only if cos $n\phi$ and $\dfrac{\partial f_{B_1}(r,\phi)}{\partial \phi}$ are not identically zero for all values of r and ϕ.

$$k_{r_2} \frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y^+_{2b}(y_{B_1})\, C_{2b} + k_{r_2} \frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y^-_{2b}(y_{B_1})\, D_{2b}$$

$$- k_{r_3} \frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)} Y^+_3(y_{B_1})\, C_3 - k_{r_3} \frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)} Y^-_3(y_{B_1})\, D_3 = 0\,, \qquad (167)$$

which is valid only if sin $n\phi$ and $\dfrac{\partial f_{B_1}(r,\phi)}{\partial r}$ are not identically zero for all values of r and ϕ.

$$J_n(k_{r_2} r)\, Y^+_{2b}(y_{B_1})\, A_{2b} + J_n(k_{r_2} r)\, Y^-_{2b}(y_{B_1})\, B_{2b}$$

$$- J_n(k_{r_3} r)\, Y^+_3(y_{B_1})\, A_3 - J_n(k_{r_3} r)\, Y^-_3(y_{B_1})\, B_3 = 0\,, \qquad (168)$$

which is valid only if $\sin n\phi$ and $\dfrac{\partial f_{B_1}(r,\phi)}{\partial \phi}$ are not identically zero for all values of r and ϕ.

$$J_n(k_{r_3} r)\frac{dY^+_3(y_{B_2})}{dy} A_3 + J_n(k_{r_3} r)\frac{dY^-_3(y_{B_2})}{dy} B_3 - J_n(k_{r_4} r)\frac{dY^+_4(y_{B_2})}{dy} A_4 = 0\,, \qquad (169)$$

which is valid only if $\cos n\phi$ is not identically zero for all values of ϕ.

$$J_n(k_{r_3} r)\frac{dY^+_3(y_{B_2})}{dy} C_3 + J_n(k_{r_3} r)\frac{dY^-_3(y_{B_2})}{dy} D_3 - J_n(k_{r_4} r)\frac{dY^+_4(y_{B_2})}{dy} B_4 = 0\,, \qquad (170)$$

which is valid only if $\sin n\phi$ is not identically zero for all values of ϕ.

$$k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^+(y_{B_2}) A_3 + k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^-(y_{B_2}) B_3$$

$$- k_{r_4} \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} Y_4^+(y_{B_2}) A_4 = 0 , \qquad (171)$$

which is valid only if cos $n\phi$ and $\frac{\partial f_{B_2}(r,\phi)}{\partial r}$ are not identically zero for all values of r and ϕ.

$$J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, C_3 + J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, D_3 - J_n(k_{r_4} r)\, Y_4^+(y_{B_2})\, B_4 = 0 , \qquad (172)$$

which is valid only if cos $n\phi$ and $\frac{\partial f_{B_2}(r,\phi)}{\partial \phi}$ are not identically zero for all values of r and ϕ.

$$k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^+(y_{B_2}) C_3 + k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^-(y_{B_2}) D_3$$

$$- k_{r_4} \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} Y_4^+(y_{B_2}) B_4 = 0 , \qquad (173)$$

which is valid only if $\sin n\phi$ and $\dfrac{\partial f_{B_2}(r,\phi)}{\partial r}$ are not identically zero for all values of r and ϕ.

$$J_n(k_{r_3} r)\, Y_3^+(y_{B_2})\, A_3 + J_n(k_{r_3} r)\, Y_3^-(y_{B_2})\, B_3 - J_n(k_{r_4} r)\, Y_4^+(y_{B_2})\, A_4 = 0\,, \quad (174)$$

which is valid only if $\sin n\phi$ and $\dfrac{\partial f_{B_2}(r,\phi)}{\partial \phi}$ are not identically zero for all values of r and ϕ.

$$\left\{\frac{dY_{2a}^+(y_0)}{dy} A_{2a} + \frac{dY_{2a}^-(y_0)}{dy} B_{2a} - \frac{dY_{2b}^+(y_0)}{dy} A_{2b} - \frac{dY_{2b}^-(y_0)}{dy} B_{2b}\right\} = G_1\,, \quad (175)$$

which is valid only if $\cos n\phi$ is not identically zero for all values of ϕ.

$$\left\{\frac{dY_{2a}^+(y_0)}{dy} C_{2a} + \frac{dY_{2a}^-(y_0)}{dy} D_{2a} - \frac{dY_{2b}^+(y_0)}{dy} C_{2b} - \frac{dY_{2b}^-(y_0)}{dy} D_{2b}\right\} = G_2\,, \quad (176)$$

which is valid only if $\sin n\phi$ is not identically zero for all values of ϕ.

G. DIFFERENCES NOTED DUE TO ARBITRARY BOUNDARY SHAPE

Before going on to verify that the set of derived general boundary condition equations reduces to a well-known and well-documented set of boundary condition equations for a very specific set of waveguide conditions, the interesting and somewhat unexpected appearance of $J_n(k_{r_i})$, i = 1, 2, 3, 4

terms in some of the general boundary condition equations needs to be discussed. These terms cannot be eliminated in the general case of arbitrarily shaped boundaries because the radial component of the wave number is not constant, and in fact, it depends upon the orientation of the local normal vector to the surface.

To show that this is true, we will begin with the specific case of a planar boundary. As shown in Figure 3, the vector wave number $\mathbf{k}$ may be resolved into its component vectors, $\mathbf{k}_r$ and $\mathbf{k}_y$ with respect to the coordinate axes r and y, that is,

$$\mathbf{k} = \mathbf{k}_r + \mathbf{k}_y . \tag{177}$$

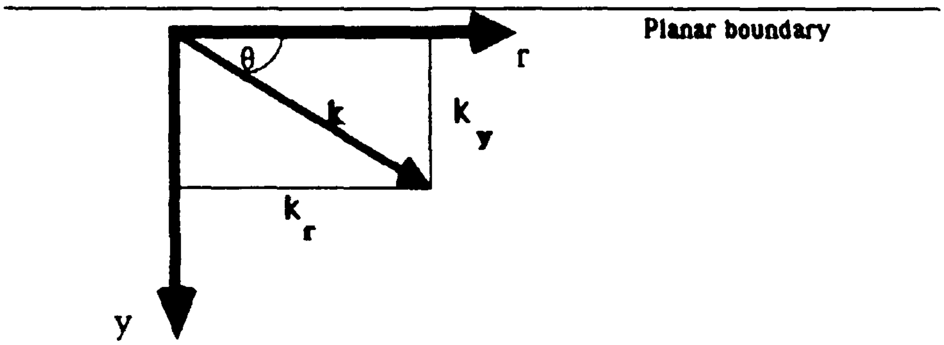

Figure 3. Planar Boundary Wave Number Vector Decomposition

Evaluating the geometry reveals the following pair of equations to describe these component vectors:

$$\mathbf{k}_r = k \cos\theta\, \hat{\mathbf{r}} = k_r \hat{\mathbf{r}} , \tag{178}$$

and

$$\mathbf{k}_y = k \sin\theta\, \hat{\mathbf{y}} = k_y \hat{\mathbf{y}} , \tag{179}$$

where k is the magnitude of the vector $\mathbf{k}$, θ is the angle between the vector $\mathbf{k}$ and the r-axis, $\hat{\mathbf{r}}$ is the unit vector in the radial direction, $\hat{\mathbf{y}}$ is the unit vector in the y direction, k_r is the magnitude of the radial component, and k_y is the magnitude of the depth component.

Now we will explore the more general case shown in Figure 4. Again the $\mathbf{k}$ vector may be decomposed into its component vectors with respect to two

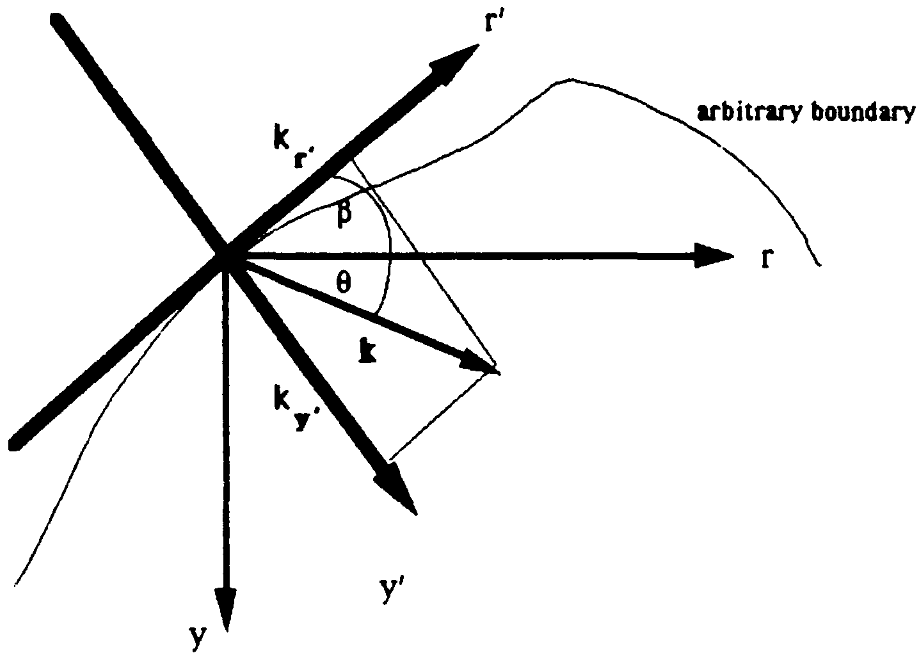

Figure 4. Generalized Boundary Wave Number Vector Decomposition

very different sets of coordinate axes, the standard r and y axes, and the r' and the y' axes, which are oriented based on the local normal vector. When decomposed with respect to the r and y axes, the components may be expressed as indicated in (178) and (179). However, when decomposed with

respect to the r′ and the y′ axes, the resulting components may be expressed as follows:

$$\mathbf{k}_{r'} = k \cos(\theta + \beta)\, \hat{\mathbf{r}}' = k_{r'}\, \hat{\mathbf{r}}' \quad , \tag{180}$$

and

$$\mathbf{k}_{y'} = k \sin(\theta + \beta)\, \hat{\mathbf{y}}' = k_{y'}\, \hat{\mathbf{y}}' \quad , \tag{181}$$

where k is the magnitude of the vector $\mathbf{k}$, θ is the angle between the vector $\mathbf{k}$ and the r-axis, β is the angle between the r and r′ axes, $\hat{\mathbf{r}}'$ is the unit vector in the r′ direction, $\hat{\mathbf{y}}'$ is the unit vector in the y′ direction, $k_{r'}$ is the magnitude of the component in the r′ direction, and $k_{y'}$ is the magnitude of the component in the y′ direction.

Using the appropriate trigonometric identities, (180) and (181) may be rewritten as

$$\mathbf{k}_{r'} = k\,(\cos\theta\, \cos\beta - \sin\theta \sin\beta)\, \hat{\mathbf{r}}' \quad , \tag{182}$$

and

$$\mathbf{k}_{y'} = k\,(\sin\theta\, \cos\beta + \cos\theta \sin\beta)\, \hat{\mathbf{y}}' \quad . \tag{183}$$

Equations (182) and (183) may be simplified further by carrying out the indicated multiplications and using (178) and (179), revealing

$$\mathbf{k}_{r'} = (k_r \cos\beta - k_y \sin\beta)\,\hat{\mathbf{r}}' \quad , \tag{184}$$

and

$$\mathbf{k}_{y'} = (k_r \sin\beta + k_y \cos\beta)\,\hat{\mathbf{y}}' \quad . \tag{185}$$

Equation (184) shows that the component of the wave number along the tangent plane (i.e., in the r' direction) at any point depends on the angle β, and therefore, on the specific point along the arbitrarily shaped boundary at which the vector is to be evaluated. Therefore, this component is not constant, and must be maintained in the boundary condition equations.

H. VERIFICATION OF INITIAL RESULTS

We will now show that the boundary condition equations derived in this section reduce to the well-known and well-documented set of boundary condition equations for the following classical waveguide problem: Assume that there are only three fluid media, not four, and that

All sound speeds are constant, i.e.,

- $c_1(y) = c_1$,
- $c_2(y) = c_2$, and
- $c_3(y) = c_3$.

All ambient densities are constant, i.e.,

- $\rho_1(y) = \rho_1$,
- $\rho_2(y) = \rho_2$, and
- $\rho_3(y) = \rho_3$.

All boundaries are planar and parallel, i.e.,

- $y_S(r,\phi) = 0$,
- $y_0(r,\phi) = y_0$, and
- $y_{B_1}(r,\phi) = D$.

These conditions represent a waveguide made up of three layers. The flat boundary at $y = 0$ separates a semi-infinite medium (medium I: $-\infty \le y \le 0$) and a finite medium (medium II: $0 \le y \le D$) of (perhaps) different specific acoustic impedances. The flat boundary at $y = D$ separates a finite medium (medium II: $0 \le y \le D$) and a semi-infinite medium (medium III: $D \le y \le +\infty$) of (perhaps) different specific acoustic impedances. These conditions imply that the following arbitrary constants may be set equal to zero for the reasons indicated:

- B_3 (no wave reflected in negative y direction)
- D_3 (no wave reflected in negative y direction)
- A_4 (medium not modeled)
- B_4 (medium not modeled)

Also, $n = 0$ since plane, parallel boundaries remove angular dependence.

These conditions also imply that the wave number k and the propagation vector component in the y direction k_y are constant in a given fluid medium. In this case, the solution to (28) is known, and can be written as follows:

$$Y(y) = A_y e^{-jk_y y} + B_y e^{+jk_y y} . \qquad (186)$$

Thus, we may set the arbitrary functions $Y^{+}(y)$ and $Y^{-}(y)$ in our previous work as follows:

$$Y^{+}(y) = A_y e^{-jk_y y}, \tag{187}$$

and

$$Y^{-}(y) = B_y e^{+jk_y y}. \tag{188}$$

Also, since all of the boundaries are plane, parallel surfaces, $\frac{\partial f_S}{\partial r}$, $\frac{\partial f_{B_1}}{\partial r}$, $\frac{\partial f_S}{\partial \phi}$, and $\frac{\partial f_{B_1}}{\partial \phi}$ are identically zero for all values of r and ϕ. Thus, the following boundary condition equations have been invalidated for the reasons indicated:

- (150) is invalid because $\sin n\phi$ (for $n = 0$) is identically zero for all values of ϕ,
- (152) is invalid because $\sin n\phi$ (for $n = 0$) is identically zero for all values of ϕ,
- (154) is invalid because $\sin n\phi$ (for $n = 0$) is identically zero for all values of ϕ.
- (155) is invalid because medium IV is not being modeled,
- (156) is invalid because medium IV is not being modeled,
- (158) is invalid because $\sin n\phi$ (for $n = 0$) is identically zero for all values of ϕ,
- (159) is invalid because $\frac{\partial f_S(r,\phi)}{\partial r}$ is identically zero for all values of r and ϕ,

• (160) is invalid because $\frac{\partial f_S(r,\phi)}{\partial \phi}$ is identically zero for all values of r and ϕ,

• (161) is invalid because both sin $n\phi$ and $\frac{\partial f_S(r,\phi)}{\partial r}$ are identically zero for all values of r and ϕ,

• (162) is invalid because both sin $n\phi$ and $\frac{\partial f_S(r,\phi)}{\partial \phi}$ are identically zero for all values of r and ϕ,

• (164) is invalid because sin $n\phi$ (for $n = 0$) is identically zero for all values of ϕ,

• (165) is invalid because $\frac{\partial f_{B_1}(r,\phi)}{\partial r}$ is identically zero for all values of r and ϕ,

• (166) is invalid because $\frac{\partial f_{B_1}(r,\phi)}{\partial \phi}$ is identically zero for all values of r and ϕ.

• (167) is invalid because both sin $n\phi$ and $\frac{\partial f_{B_1}(r,\phi)}{\partial r}$ are identically zero for all values of r and ϕ,

• 168) is invalid because both sin $n\phi$ and $\frac{\partial f_{B_1}(r,\phi)}{\partial \phi}$ are identically zero for all values of r and ϕ,

• (169) is invalid because medium IV is not being modeled,

• (170) is invalid because medium IV is not being modeled,

• (171) is invalid because medium IV is not being modeled,

• (172) is invalid because medium IV is not being modeled,

• (173) is invalid because medium IV is not being modeled,

• (174) is invalid because medium IV is not being modeled, and

• (176) is invalid because sin $n\phi$ (for $n = 0$) is identically zero for all values of ϕ.

Thus, the original set of 28 equations in 17 unknowns has been reduced to a set of six equations in six unknowns. The next step will be to evaluate each of the remaining equations in turn so that they may be compared with the equations developed by Ziomek (1991) for this particular waveguide problem.

The first of the remaining equations is

$$\rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^{+}(y_S)\, A_{2a} + \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^{-}(y_S)\, B_{2a}$$
$$- \rho_1(y_S)\, J_n(k_{r_1} r)\, Y_1^{-}(y_S)\, B_1 = 0 \, . \qquad (149)$$

The Bessel function dependence of (149) may be eliminated by virtue of the fact that, in this simple waveguide problem, the radial component of the propagation vector is the same in *all three media* (*implying* that the Bessel functions may just be divided out). Recalling that the densities are constants and that the value of y at y_S is identically zero, (149) becomes

$$\rho_2\, Y_{2a}^{+}(0)\, A_{2a} + \rho_2\, Y_{2a}^{-}(0)\, B_{2a} - \rho_1\, Y_1^{-}(0)\, B_1 = 0 \, . \qquad (189)$$

Substituting the appropriate functions of y into the simplified expression (189), and noting that e^{i0} is equal to unity yields the following:

$$\rho_2\, A_{2a} + \rho_2\, B_{2a} - \rho_1\, B_1 = 0 \, . \qquad (190)$$

This equation is the same as that derived by Ziomek (1991, equation (3.9-25)) for this boundary condition.

Conducting a similar analysis on (151) yields

$$e^{-jk_{y_2} y_0} A_{2a} + e^{+jk_{y_2} y_0} B_{2a} - e^{-jk_{y_2} y_0} A_{2b} - e^{+jk_{y_2} y_0} B_{2b} = 0 . \qquad (191)$$

This equation is the same as that derived by Ziomek (1991, equation (3.9-34)) for this boundary condition.

Equation (153) reduces to the following (after additionally noting that B_3 has been set equal to zero)

$$\rho_2 e^{-jk_{y_2} D} A_{2b} + \rho_2 e^{+jk_{y_2} D} B_{2b} - \rho_3 e^{-jk_{y_3} D} A_3 = 0 . \qquad (192)$$

This equation is the same as that derived by Ziomek (1991, equation (3.9-44)) for this boundary condition.

Equation (157) becomes

$$k_{y_2} A_{2a} - k_{y_2} B_{2a} + k_{y_1} B_1 = 0 . \qquad (193)$$

This equation is the same as that derived by Ziomek (1991, equation (3.9-30)) for this boundary condition.

Equation (163) reduces to the following (after additionally noting that B_3 has been set equal to zero):

$$k_{y_2} e^{-jk_{y_2} D} A_{2b} - k_{y_2} e^{+jk_{y_2} D} B_{2b} - k_{y_3} e^{-jk_{y_3} D} A_3 = 0 . \qquad (194)$$

This equation is the same as that derived by Ziomek (1991, equation (3.9-49)) for this boundary condition.

In order to evaluate (175), let $G_1 = \frac{-k_r}{2\pi}$ (as suggested by Ziomek (1991, equation (3.9-35) and following)). Substituting yields

$$- j k_{y_2} e^{-jk_{y_2} y_0} A_{2a} + j k_{y_2} e^{+jk_{y_2} y_0} B_{2a}$$

$$+ j k_{y_2} e^{-jk_{y_2} y_0} A_{2b} - j k_{y_2} e^{+jk_{y_2} y_0} B_{2b} = \frac{-k_r}{2\pi} . \qquad (195)$$

This reduces to

$$- e^{-jk_{y_2} y_0} A_{2a} + e^{+jk_{y_2} y_0} B_{2a} + e^{-jk_{y_2} y_0} A_{2b} - e^{+jk_{y_2} y_0} B_{2b} = + j \frac{k_r}{2\pi \ k_{y_2}} . \qquad (196)$$

This equation is the same as that derived by Ziomek (1991, equation (3.9-38)) for this boundary condition.

Thus, we have shown that the theoretically derived set of equations for a general waveguide problem reduces to the set of equations expected for the classical waveguide problem. This provides us with the confidence to go on with the solution for the unknown arbitrary constants.

IV. SOLUTION FOR THE UNKNOWN ARBITRARY CONSTANTS USING SYMBOLIC ALGEBRA CAPABILITIES OF *Mathematica*

In Section III of this thesis, we developed a set of 28 boundary condition equations in the 17 unknown constants. The purpose of this section is to generate a solution to this system of equations for the general waveguide case.

A review of (149) through (176) reveals that the coefficients of these unknown constants are, in general, complicated expressions involving depth-dependent densities, range-dependent n^{th} order Bessel functions, and as yet unspecified depth-dependent velocity potential functions (i.e., the Y_x^+ and Y_x^- terms). In order to maintain the generality of the generated solution, we will require either many long hours of tedious algebra involving manipulations of these complicated expressions (with the high probability of algebraic errors) or a computer program capable of conducting such manipulations directly on these symbolic expressions. Fortunately, *Mathematica* for the Macintosh computer (version 1.2.1 f33 (enhanced)) is the one such program available to us at the Naval Postgraduate School, and therefore will be used to generate the general solution desired.

The first step in this process will be to program *Mathematica* to solve for the unknown constants for a very specific set of waveguide conditions. By doing this, we will gain experience in using the program and confidence that the program output is reliable. For this work, we will use a three media

waveguide with plane, parallel boundaries. Using vector-matrix notation, a compact system equation may be written for this (or for any other) case as follows:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}\,, \tag{198}$$

where **A** is the matrix of coefficients, **x** is the column vector of unknown constants, and **b** is the column vector of known constants.

For the three media waveguide with plane, parallel boundaries, these vector-matrix quantities are defined as follows:

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & 0 & 0 & 0 \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & 0 \\ 0 & 0 & 0 & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{5,4} & a_{5,5} & a_{5,6} \\ 0 & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & 0 \end{bmatrix}$$

where

$$a_{1,1} = -\,\rho_1(y_S)\,Y_1^-(y_S) \tag{199}$$

$$a_{1,2} = \rho_2(y_S)\,Y_{2a}^+(y_S) \tag{200}$$

$$a_{1,3} = \rho_2(y_S)\,Y_{2a}^-(y_S) \tag{201}$$

$$a_{2,2} = Y_{2a}^+(y_o) \tag{202}$$

$$a_{2,3} = Y_{2a}^-(y_o) \tag{203}$$

$$a_{2,4} = -\,Y_{2b}^+(y_o) \tag{204}$$

$$a_{2,5} = -\,Y_{2b}^-(y_o) \tag{205}$$

$$a_{3,4} = \rho_2(y_{B_1})\, Y_{2b}^{+}(y_{B_1}) \tag{206}$$

$$a_{3,5} = \rho_2(y_{B_1})\, Y_{2b}^{-}(y_{B_1}) \tag{207}$$

$$a_{3,6} = -\rho_3(y_{B_1})\, Y_3^{+}(y_{B_1}) \tag{208}$$

$$a_{4,1} = -\frac{dY_1^{-}(y_S)}{dy} \tag{209}$$

$$a_{4,2} = \frac{dY_{2a}^{+}(y_S)}{dy} \tag{210}$$

$$a_{4,3} = \frac{dY_{2a}^{-}(y_S)}{dy} \tag{211}$$

$$a_{5,4} = \frac{dY_{2b}^{+}(y_{B_1})}{dy} \tag{212}$$

$$a_{5,5} = \frac{dY_{2b}^{-}(y_{B_1})}{dy} \tag{213}$$

$$a_{5,6} = -\frac{dY_3^{+}(y_{B_1})}{dy} \tag{214}$$

$$a_{6,2} = \frac{dY_{2a}^{+}(y_o)}{dy} \tag{215}$$

$$a_{6,3} = \frac{dY_{2a}^{-}(y_o)}{dy} \tag{216}$$

$$a_{6,4} = -\frac{dY_{2b}^{+}(y_o)}{dy} \tag{217}$$

$$a_{6,5} = -\frac{dY_{2b}^{-}(y_o)}{dy};\tag{218}$$

$$\mathbf{x} = [B_1 \;\; A_{2a} \;\; B_{2a} \;\; A_{2b} \;\; B_{2b} \;\; A_3]^T\tag{219}$$

where the superscript T indicates the transpose matrix operator (indicating that $\mathbf{x}$ is a column vector),

$$\mathbf{b} = [0\ 0\ 0\ 0\ 0\ G_1]^T\tag{220}$$

where the superscript T indicates that $\mathbf{b}$ is a column vector, and

$$G_1 = -\frac{k_r}{2\pi}.\tag{221}$$

It should be noted here that we have defined the matrix $\mathbf{A}$ in a very specific manner. Each row of $\mathbf{A}$ represents one of the valid boundary condition equations for the specific waveguide being studied. These appear in the order presented in Section III. For the three media waveguide with plane, parallel boundaries, row 1 of $\mathbf{A}$ contains the coefficients found in (149). Row 2 contains the coefficients found in (151), and so on. Row 6 contains the coefficients found in (175). For simplicity's sake, we have used generic elements, such as $a_{1,2}$, to replace the more complicated expressions. In addition, zeros have been used to indicate that the appropriate unknown constants do not appear in a specific boundary condition equation. We will use this convention in the work which follows.

In this three media waveguide case, **A** is a six by six square matrix. Therefore, the solution to (198) may be written directly as

$$\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{b}\,. \tag{222}$$

where the superscript -1 indicates the inverse matrix operator.

Using Wolfram (1988) as a programming reference guide, a *Mathematica* "notebook" was created to solve this three media waveguide case using the solution technique expressed in (222). The *Mathematica* code required to perform this task is as follows:

```
a = {{a1c1, a1c2, a1c3, 0, 0, 0},
     {0, a2c2, a2c3, a2c4, a2c5, 0},
     {0, 0, 0, a3c4, a3c5, a3c6},
     {a4c1, a4c2, a4c3, 0, 0, 0},
     {0, 0, 0, a5c4, a5c5, a5c6},
     {0, a6c2, a6c3, a6c4, a6c5, 0}};

b = {0, 0, 0, 0, 0, G1};

x = (Inverse[a]).b
```

In developing this code, we continued to utilize the generic matrix elements described earlier. Two subtle differences in the notation used in the code from the notation discussed earlier need to be pointed out. First, we have represented the elements of matrix **A** (for example, $a_{1,2}$) as individual variables (the corresponding variable name would be a1c2). This slight deviation in notation was used because subscripting as defined by *Mathematica* would not have been useful for our purposes. In this revised notation, the small case letter "c" represents the *comma* in the element name.

This deviation was required because the program uses the comma to separate individual array elements. The second notational comment refers to the fact that a lower case letter "a" was used to represent the matrix **A**. This was required to conform with *Mathematica* 's notational convention, which reserves names beginning with capital letters for built-in functions.

Unfortunately, running this code resulted in halted execution due to a singularity error. We surmise that the problem occurred when the program was attempting to take the inverse of the matrix **A**. Luckily, *Mathematica* has a built-in function, LinearSolve, which evaluates (198) directly if the matrix **A** is a square matrix. Thus, the following revised code was written:

```
a = {{a1c1, a1c2, a1c3, 0, 0, 0},
     {0, a2c2, a2c3, a2c4, a2c5, 0},
     {0, 0, 0, a3c4, a3c5, a3c6},
     {a4c1, a4c2, a4c3, 0, 0, 0},
     {0, 0, 0, a5c4, a5c5, a5c6},
     {0, a6c2, a6c3, a6c4, a6c5, 0}};

b = {0, 0, 0, 0, 0, G1};

LinearSolve[a,b]
```

The revised code ran successfully. The output of this code is the desired vector **x**. When *Mathematica* functions such as Factor, Cancel, and Simplify were applied to the output, the same result was returned, indicating that the program was satisfied that the output was as simple as it could make it. Closer inspection of the output revealed that each of the six elements was of the form

$$x_1 = \frac{num_1}{denom}. \tag{223}$$

where x_1 represents the first element of the vector x (in the three media waveguide with plane, parallel boundaries, this element is the unknown constant B_1), num_1 represents the numerator expression for the first element, and denom represents the denominator. Fortunately, all of the elements of the output vector have a common denominator. This inspection also revealed that some algebraic manipulations could be manually performed to simplify the expressions somewhat. Thus, the robustness of the symbolic algebra functions of *Mathematica* is at best questionable.

We will now present the results of the program for the three media waveguide with plane, parallel boundaries. The first step will be to simplify the results manually in order to generate generic expressions for the unknown constants in terms of the generic elements. Second, we will substitute (199) through (218) and (221) into the generic expressions to reveal general expressions for these unknown constants. Finally, we will assume constant speed of sound and constant density and show that the general expressions formed from the *Mathematica* output are the same as those derived by Ziomek (1991) for the classical waveguide case. This verification will be conducted in the following order: A_{2a}, B_{2a}, A_{2b}, B_{2b}, B_1, and A_3. We will demonstrate this entire process for the unknown constant A_{2a} only, and simply present the results for the other five unknown constants.

The first output expression to be explored will be the common denominator, denom

$$
\begin{aligned}
\text{denom} = & - a_{1,3}\, a_{2,5}\, a_{3,6}\, a_{4,1}\, a_{5,4}\, a_{6,2} + a_{1,1}\, a_{2,5}\, a_{3,6}\, a_{4,3}\, a_{5,4}\, a_{6,2} \\
& + a_{1,3}\, a_{2,4}\, a_{3,6}\, a_{4,1}\, a_{5,5}\, a_{6,2} - a_{1,1}\, a_{2,4}\, a_{3,6}\, a_{4,3}\, a_{5,5}\, a_{6,2} \\
& + a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,1}\, a_{5,6}\, a_{6,2} - a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,1}\, a_{5,6}\, a_{6,2} \\
& - a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,3}\, a_{5,6}\, a_{6,2} + a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,3}\, a_{5,6}\, a_{6,2} \\
& + a_{1,2}\, a_{2,5}\, a_{3,6}\, a_{4,1}\, a_{5,4}\, a_{6,3} - a_{1,1}\, a_{2,5}\, a_{3,6}\, a_{4,2}\, a_{5,4}\, a_{6,3} \\
& - a_{1,2}\, a_{2,4}\, a_{3,6}\, a_{4,1}\, a_{5,5}\, a_{6,3} + a_{1,1}\, a_{2,4}\, a_{3,6}\, a_{4,2}\, a_{5,5}\, a_{6,3} \\
& - a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,1}\, a_{5,6}\, a_{6,3} + a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,1}\, a_{5,6}\, a_{6,3} \\
& + a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,2}\, a_{5,6}\, a_{6,3} - a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,2}\, a_{5,6}\, a_{6,3} \\
& - a_{1,3}\, a_{2,2}\, a_{3,6}\, a_{4,1}\, a_{5,5}\, a_{6,4} + a_{1,2}\, a_{2,3}\, a_{3,6}\, a_{4,1}\, a_{5,5}\, a_{6,4} \\
& - a_{1,1}\, a_{2,3}\, a_{3,6}\, a_{4,2}\, a_{5,5}\, a_{6,4} + a_{1,1}\, a_{2,2}\, a_{3,6}\, a_{4,3}\, a_{5,5}\, a_{6,4} \\
& + a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,1}\, a_{5,6}\, a_{6,4} - a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,1}\, a_{5,6}\, a_{6,4} \\
& + a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,2}\, a_{5,6}\, a_{6,4} - a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,3}\, a_{5,6}\, a_{6,4} \\
& + a_{1,3}\, a_{2,2}\, a_{3,6}\, a_{4,1}\, a_{5,4}\, a_{6,5} - a_{1,2}\, a_{2,3}\, a_{3,6}\, a_{4,1}\, a_{5,4}\, a_{6,5} \\
& + a_{1,1}\, a_{2,3}\, a_{3,6}\, a_{4,2}\, a_{5,4}\, a_{6,5} - a_{1,1}\, a_{2,2}\, a_{3,6}\, a_{4,3}\, a_{5,4}\, a_{6,5} \\
& - a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,1}\, a_{5,6}\, a_{6,5} + a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,1}\, a_{5,6}\, a_{6,5} \\
& - a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,2}\, a_{5,6}\, a_{6,5} + a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,3}\, a_{5,6}\, a_{6,5}\,.
\end{aligned}
\tag{224}
$$

Factoring (224) reveals

$$
\begin{aligned}
\text{denom} = & \; a_{1,1}\, a_{2,5}\, a_{4,3}\, a_{6,2}\, (a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}) \\
& - a_{1,3}\, a_{2,5}\, a_{4,1}\, a_{6,2}\, (a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}) \\
& + a_{1,3}\, a_{2,4}\, a_{4,1}\, a_{6,2}\, (a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \\
& - a_{1,1}\, a_{2,4}\, a_{4,3}\, a_{6,2}\, (a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \\
& + a_{1,2}\, a_{2,5}\, a_{4,1}\, a_{6,3}\, (a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}) \\
& - a_{1,1}\, a_{2,5}\, a_{4,2}\, a_{6,3}\, (a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}) \\
& + a_{1,1}\, a_{2,4}\, a_{4,2}\, a_{6,3}\, (a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \\
& - a_{1,2}\, a_{2,4}\, a_{4,1}\, a_{6,3}\, (a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \\
& + a_{1,2}\, a_{2,3}\, a_{4,1}\, a_{6,4}\, (a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \\
& - a_{1,3}\, a_{2,2}\, a_{4,1}\, a_{6,4}\, (a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \\
& + a_{1,1}\, a_{2,2}\, a_{4,3}\, a_{6,4}\, (a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \\
& - a_{1,1}\, a_{2,3}\, a_{4,2}\, a_{6,4}\, (a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \\
& + a_{1,3}\, a_{2,2}\, a_{4,1}\, a_{6,5}\, (a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}) \\
& - a_{1,2}\, a_{2,3}\, a_{4,1}\, a_{6,5}\, (a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6})
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{1,1}\, a_{2,3}\, a_{4,2}\, a_{6,5} \{a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}\} \\
&- a_{1,1}\, a_{2,2}\, a_{4,3}\, a_{6,5} \{a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}\} .
\end{aligned}
\tag{225}
$$

Collecting common terms yields the following *generic expression* for the *common denominator*:

$$
\begin{aligned}
\text{denom} = {} & \{a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}\} \, [a_{1,1}\, a_{2,5}\, a_{4,3}\, a_{6,2} \\
& - a_{1,3}\, a_{2,5}\, a_{4,1}\, a_{6,2} + a_{1,2}\, a_{2,5}\, a_{4,1}\, a_{6,3} - a_{1,1}\, a_{2,5}\, a_{4,2}\, a_{6,3} \\
& + a_{1,3}\, a_{2,2}\, a_{4,1}\, a_{6,5} - a_{1,2}\, a_{2,3}\, a_{4,1}\, a_{6,5} + a_{1,1}\, a_{2,3}\, a_{4,2}\, a_{6,5} \\
& - a_{1,1}\, a_{2,2}\, a_{4,3}\, a_{6,5}] + \{a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}\} \, [a_{1,3}\, a_{2,4}\, a_{4,1}\, a_{6,2} \\
& - a_{1,1}\, a_{2,4}\, a_{4,3}\, a_{6,2} + a_{1,1}\, a_{2,4}\, a_{4,2}\, a_{6,3} - a_{1,2}\, a_{2,4}\, a_{4,1}\, a_{6,3} \\
& + a_{1,2}\, a_{2,3}\, a_{4,1}\, a_{6,4} - a_{1,3}\, a_{2,2}\, a_{4,1}\, a_{6,4} + a_{1,1}\, a_{2,2}\, a_{4,3}\, a_{6,4} \\
& - a_{1,1}\, a_{2,3}\, a_{4,2}\, a_{6,4}] .
\end{aligned}
\tag{226}
$$

Substituting (199) through (218) into (226), and using the facts that $y_S = 0$ and $y_{B_1} = D$ reveals

$$
\begin{aligned}
\text{denom} = {} & \left\{\rho_2(D)\, Y_{2b}^{+}(D) \frac{dY_3^{+}(D)}{dy} - \rho_3(D) \frac{dY_{2b}^{+}(D)}{dy} Y_3^{+}(D)\right\} \\
& \times \left[\rho_1(0)\, Y_1^{-}(0) \frac{dY_{2a}^{+}(y_o)}{dy} \frac{dY_{2a}^{-}(0)}{dy} Y_{2b}^{-}(y_o) - \rho_2(0) \frac{dY_1^{-}(0)}{dy} \frac{dY_{2a}^{+}(y_o)}{dy} Y_{2a}^{-}(0)\, Y_{2b}^{-}(y_o)\right. \\
& + \rho_2(0) \frac{dY_1^{-}(0)}{dy} Y_{2a}^{+}(0) \frac{dY_{2a}^{-}(y_o)}{dy} Y_{2b}^{-}(y_o) - \rho_1(0)\, Y_1^{-}(0) \frac{dY_{2a}^{+}(0)}{dy} \frac{dY_{2a}^{-}(y_o)}{dy} Y_{2b}^{-}(y_o)
\end{aligned}
$$

$$+ \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(y_o) Y_{2a}^-(0) \frac{dY_{2b}^-(y_o)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(0) Y_{2a}^-(y_o) \frac{dY_{2b}^-(y_o)}{dy}$$

$$+ \rho_1(0) Y_1^-(0) \frac{dY_{2a}^+(0)}{dy} Y_{2a}^-(y_o) \frac{dY_{2b}^-(y_o)}{dy} - \rho_1(0) Y_1^-(0) Y_{2a}^+(y_o) \frac{dY_{2a}^-(0)}{dy} \frac{dY_{2b}^-(y_o)}{dy} \Big]$$

$$+ \Big\{ \rho_2(D) Y_{2b}^-(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^-(D)}{dy} Y_3^+(D) \Big\}$$

$$\times \Big[\rho_2(0) \frac{dY_1^-(0)}{dy} \frac{dY_{2a}^+(y_o)}{dy} Y_{2a}^-(0) Y_{2b}^+(y_o) - \rho_1(0) Y_1^-(0) \frac{dY_{2a}^+(y_o)}{dy} \frac{dY_{2a}^-(0)}{dy} Y_{2b}^+(y_o)$$

$$+ \rho_1(0) Y_1^-(0) \frac{dY_{2a}^+(0)}{dy} \frac{dY_{2a}^-(y_o)}{dy} Y_{2b}^+(y_o) - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(0) \frac{dY_{2a}^-(y_o)}{dy} Y_{2b}^+(y_o)$$

$$+ \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(0) Y_{2a}^-(y_o) \frac{dY_{2b}^+(y_o)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(y_o) Y_{2a}^-(0) \frac{dY_{2b}^+(y_o)}{dy}$$

$$+ \rho_1(0) Y_1^-(0) Y_{2a}^+(y_o) \frac{dY_{2a}^-(0)}{dy} \frac{dY_{2b}^+(y_o)}{dy} - \rho_1(0) Y_1^-(0) \frac{dY_{2a}^+(0)}{dy} Y_{2a}^-(y_o) \frac{dY_{2b}^+(y_o)}{dy} \Big] . \tag{227}$$

Simplifying (227) reveals

$$
\begin{aligned}
\text{denom} = \Big[& \rho_1(0)\, Y_1^-(0) \left(\frac{dY_{2a}^+(y_o)}{dy} \frac{dY_{2a}^-(0)}{dy} - \frac{dY_{2a}^+(0)}{dy} \frac{dY_{2a}^-(y_o)}{dy} \right) \\
& - \rho_2(0) \frac{dY_1^-(0)}{dy} \left(\frac{dY_{2a}^+(y_o)}{dy} Y_{2a}^-(0) - Y_{2a}^+(0) \frac{dY_{2a}^-(y_o)}{dy} \right) \Big] \\
\times \Big[& Y_{2b}^-(y_o) \left\{ \rho_2(D)\, Y_{2b}^+(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^+(D)}{dy} Y_3^+(D) \right\} \\
& - Y_{2b}^+(y_o) \left\{ \rho_2(D)\, Y_{2b}^-(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^-(D)}{dy} Y_3^+(D) \right\} \Big] \\
- \Big[& \rho_1(0)\, Y_1^-(0) \left(Y_{2a}^+(y_o) \frac{dY_{2a}^-(0)}{dy} - \frac{dY_{2a}^+(0)}{dy} Y_{2a}^-(y_o) \right) \\
& - \rho_2(0) \frac{dY_1^-(0)}{dy} \left(Y_{2a}^+(y_o)\, Y_{2a}^-(0) - Y_{2a}^+(0)\, Y_{2a}^-(y_o) \right) \Big] \\
\times \Big[& \frac{dY_{2b}^-(y_o)}{dy} \left\{ \rho_2(D)\, Y_{2b}^+(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^+(D)}{dy} Y_3^+(D) \right\} \\
& - \frac{dY_{2b}^+(y_o)}{dy} \left\{ \rho_2(D)\, Y_{2b}^-(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^-(D)}{dy} Y_3^+(D) \right\} \Big] .
\end{aligned}
\tag{228}
$$

Equation (228) is the final *general* form of the *Mathematica* output *common denominator*. If we now assume constant speed of sound and constant density in a specific medium, the depth-dependent functions in (228) become complex exponentials and the denominator becomes (*using the subscript* c *to indicate the constant speed of sound assumption*)

$$\mathrm{denom}_c = -j\,2\,k_{y_2}\,e^{-jk_{y_3}D}$$

$$\times\Big\{(\rho_1 k_{y_2} - \rho_2 k_{y_1})(\rho_2 k_{y_3} - \rho_3 k_{y_2})\,e^{-jk_{y_2}D}$$

$$+(\rho_1 k_{y_2} + \rho_2 k_{y_1})(\rho_2 k_{y_3} + \rho_3 k_{y_2})\,e^{+jk_{y_2}D}\Big\}. \qquad (229)$$

Now that the denominator has been simplified, we'll concentrate on obtaining expressions for each of the unknown constants in the order stated above. The first constant is

$$A_{2a} = \frac{\mathrm{num}_2}{\mathrm{denom}}, \qquad (230)$$

where

$$\begin{aligned}\mathrm{num}_2 = G_1 \big[&- a_{1,3}\,a_{2,5}\,a_{3,6}\,a_{4,1}\,a_{5,4} + a_{1,1}\,a_{2,5}\,a_{3,6}\,a_{4,3}\,a_{5,4} \\ &+ a_{1,3}\,a_{2,4}\,a_{3,6}\,a_{4,1}\,a_{5,5} - a_{1,1}\,a_{2,4}\,a_{3,6}\,a_{4,3}\,a_{5,5} \\ &+ a_{1,3}\,a_{2,5}\,a_{3,4}\,a_{4,1}\,a_{5,6} - a_{1,3}\,a_{2,4}\,a_{3,5}\,a_{4,1}\,a_{5,6} \\ &- a_{1,1}\,a_{2,5}\,a_{3,4}\,a_{4,3}\,a_{5,6} + a_{1,1}\,a_{2,4}\,a_{3,5}\,a_{4,3}\,a_{5,6}\big]. \end{aligned} \qquad (231)$$

Factoring (231) and collecting common terms yields the following *generic expression* for the *numerator* of A_{2a}:

$$\begin{aligned} num_2 = G_1 \Big[& (a_{1,1}\, a_{2,5}\, a_{4,3} - a_{1,3}\, a_{2,5}\, a_{4,1}) (a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}) \\ & + (a_{1,1}\, a_{2,4}\, a_{4,3} - a_{1,3}\, a_{2,4}\, a_{4,1}) (a_{3,5}\, a_{5,6} - a_{3,6}\, a_{5,5}) \Big] \,. \end{aligned} \tag{232}$$

Substituting appropriate expressions into (232) yields the following *general expression* for the *numerator* of A_{2a}:

$$\begin{aligned} num_2 = \frac{-k_r}{2\pi} & \left(\rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^-(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^-(0) \right) \\ \times & \Big[Y_{2b}^-(y_o) \left\{ \rho_2(D)\, Y_{2b}^+(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^+(D)}{dy} Y_3^+(D) \right\} \\ - & Y_{2b}^+(y_o) \left\{ \rho_2(D)\, Y_{2b}^-(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^-(D)}{dy} Y_3^+(D) \right\} \Big] \,. \end{aligned} \tag{233}$$

Again, making the constant speed of sound and constant density assumptions and substituting the appropriate depth-dependent expressions allows us to write the numerator of A_{2a} as (*using the subscript* c *to indicate the constant speed of sound assumption*)

$$num_{2c} = \frac{k_r}{2\pi}(\rho_1 k_{y_2} - \rho_2 k_{y_1}) e^{-jk_{y_3} D}$$

$$\times \left[(\rho_3 k_{y_2} - \rho_2 k_{y_3}) e^{+jk_{y_2} y_o} e^{-jk_{y_2} D} + (\rho_2 k_{y_3} + \rho_3 k_{y_2}) e^{-jk_{y_2} y_o} e^{+jk_{y_2} D}\right].$$

(234)

Thus, the *general result* for A_{2a} is

$$A_{2a} = \frac{-k_r}{2\pi}\left(\rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^-(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^-(0)\right)$$

$$\times \left[Y_{2b}^-(y_o)\left\{\rho_2(D)\, Y_{2b}^+(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^+(D)}{dy} Y_3^+(D)\right\}\right.$$

$$\left. - Y_{2b}^+(y_o)\left\{\rho_2(D)\, Y_{2b}^-(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^-(D)}{dy} Y_3^+(D)\right\}\right] /$$

$$\left[\rho_1(0)\, Y_1^-(0)\left(\frac{dY_{2a}^+(y_o)}{dy}\frac{dY_{2a}^-(0)}{dy} - \frac{dY_{2a}^+(0)}{dy}\frac{dY_{2a}^-(y_o)}{dy}\right)\right.$$

$$\left. - \rho_2(0) \frac{dY_1^-(0)}{dy}\left(\frac{dY_{2a}^+(y_o)}{dy} Y_{2a}^-(0) - Y_{2a}^+(0) \frac{dY_{2a}^-(y_o)}{dy}\right)\right]$$

$$\times \left[Y_{2b}^{-}(y_o) \left\{ \rho_2(D)\, Y_{2b}^{+}(D) \frac{dY_3^{+}(D)}{dy} - \rho_3(D) \frac{dY_{2b}^{+}(D)}{dy} Y_3^{+}(D) \right\} \right.$$

$$\left. - Y_{2b}^{+}(y_o) \left\{ \rho_2(D)\, Y_{2b}^{-}(D) \frac{dY_3^{+}(D)}{dy} - \rho_3(D) \frac{dY_{2b}^{-}(D)}{dy} Y_3^{+}(D) \right\} \right]$$

$$- \left[\rho_1(0)\, Y_1^{-}(0) \left(Y_{2a}^{+}(y_o) \frac{dY_{2a}^{-}(0)}{dy} - \frac{dY_{2a}^{+}(0)}{dy} Y_{2a}^{-}(y_o) \right) \right.$$

$$\left. - \rho_2(0) \frac{dY_1^{-}(0)}{dy} \left(Y_{2a}^{+}(y_o)\, Y_{2a}^{-}(0) - Y_{2a}^{+}(0)\, Y_{2a}^{-}(y_o) \right) \right]$$

$$\times \left[\frac{dY_{2b}^{-}(y_o)}{dy} \left\{ \rho_2(D)\, Y_{2b}^{+}(D) \frac{dY_3^{+}(D)}{dy} - \rho_3(D) \frac{dY_{2b}^{+}(D)}{dy} Y_3^{+}(D) \right\} \right.$$

$$\left. - \frac{dY_{2b}^{+}(y_o)}{dy} \left\{ \rho_2(D)\, Y_{2b}^{-}(D) \frac{dY_3^{+}(D)}{dy} - \rho_3(D) \frac{dY_{2b}^{-}(D)}{dy} Y_3^{+}(D) \right\} \right] . \qquad (235)$$

For constant speed of sound and ambient density, A_{2a} reduces to

$$A_{2a_c} = \frac{k_r}{2\pi}\left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) e^{-jk_{y_3} D}$$

$$\times \left[\left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) e^{+jk_{y_2} y_o} e^{-jk_{y_2} D} + \left(\rho_2 k_{y_3} + \rho_3 k_{y_2}\right) e^{-jk_{y_2} y_o} e^{+jk_{y_2} D} \right] /$$

$$- j\, 2\, k_{y_2} e^{-jk_{y_3} D} \left\{ \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right)\left(\rho_2 k_{y_3} - \rho_3 k_{y_2}\right) e^{-jk_{y_2} D}\right.$$

$$\left. + \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right)\left(\rho_2 k_{y_3} + \rho_3 k_{y_2}\right) e^{+jk_{y_2} D} \right\}. \qquad (236)$$

We must now verify that (236) simplifies to the well-known and well-documented expression for this unknown constant. Eliminating the common term $e^{-jk_{y_3} D}$ and dividing numerator and denominator by $\left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right)$ yields

$$A_{2a_c} = \frac{j\, k_r}{4\pi\, k_{y_2}} \frac{(\rho_1\, k_{y_2} - \rho_2\, k_{y_1})}{(\rho_1\, k_{y_2} + \rho_2\, k_{y_1})} \left[(\rho_3\, k_{y_2} - \rho_2\, k_{y_3})\, e^{+jk_{y_2} y_o}\, e^{-jk_{y_2} D} \right.$$

$$\left. + (\rho_2\, k_{y_3} + \rho_3\, k_{y_2})\, e^{-jk_{y_2} y_o}\, e^{+jk_{y_2} D} \right] /$$

$$\left[\frac{(\rho_1\, k_{y_2} - \rho_2\, k_{y_1})}{(\rho_1\, k_{y_2} + \rho_2\, k_{y_1})} (\rho_2\, k_{y_3} - \rho_3\, k_{y_2})\, e^{-jk_{y_2} D} \right.$$

$$\left. + (\rho_2\, k_{y_3} + \rho_3\, k_{y_2})\, e^{+jk_{y_2} D} \right]. \qquad (237)$$

If we define a reflection coefficient at the boundary between medium two and medium one as suggested by Ziomek (1991, equation (3.9-54)) as follows:

$$R_{21} = \frac{(\rho_1\, k_{y_2} - \rho_2\, k_{y_1})}{(\rho_1\, k_{y_2} + \rho_2\, k_{y_1})}, \qquad (238)$$

then, (237) becomes

$$A_{2a_c} = \frac{j\,k_r}{4\pi\,k_{y_2}} R_{21} \left[(\rho_3 k_{y_2} - \rho_2 k_{y_3}) e^{+jk_{y_2} y_o} e^{-jk_{y_2} D} + \right.$$

$$\left. (\rho_2 k_{y_3} + \rho_3 k_{y_2}) e^{-jk_{y_2} y_o} e^{+jk_{y_2} D} \right] /$$

$$\left[R_{21} (\rho_2 k_{y_3} - \rho_3 k_{y_2}) e^{-jk_{y_2} D} + (\rho_2 k_{y_3} + \rho_3 k_{y_2}) e^{+jk_{y_2} D} \right]. \qquad (239)$$

Rearranging the denominator and dividing both numerator and denominator of (239) by $(\rho_2 k_{y_3} + \rho_3 k_{y_2})$ reveals

$$A_{2a_c} = \frac{j\,k_r}{4\pi\,k_{y_2}} R_{21}$$

$$\times \left[\frac{(\rho_3\, k_{y_2} - \rho_2\, k_{y_3})}{(\rho_2\, k_{y_3} + \rho_3\, k_{y_2})} e^{+jk_{y_2} y_o} e^{-jk_{y_2} D} + e^{-jk_{y_2} y_o} e^{+jk_{y_2} D} \right] /$$

$$\left[e^{+jk_{y_2}D} - R_{21}\frac{(\rho_3\ k_{y_2} - \rho_2\ k_{y_3})}{(\rho_2\ k_{y_3} + \rho_3\ k_{y_2})} e^{-jk_{y_2}D}\right]. \tag{240}$$

If we define a reflection coefficient at the boundary between medium two and medium three as suggested by Ziomek (1991, equation (3.9-55)) as follows:

$$R_{23} = \frac{(\rho_3\ k_{y_2} - \rho_2\ k_{y_3})}{(\rho_3\ k_{y_2} + \rho_2\ k_{y_3})}, \tag{241}$$

then (240) becomes

$$A_{2a_c} = \frac{j\ k_r}{4\pi\ k_{y_2}} R_{21}\left[R_{23}\, e^{+jk_{y_2}y_o}\, e^{-jk_{y_2}D} + e^{-jk_{y_2}y_o}\, e^{+jk_{y_2}D}\right] /$$

$$\left[e^{+jk_{y_2}D} - R_{21}\, R_{23}\, e^{-jk_{y_2}D}\right]. \tag{242}$$

Dividing the numerator and denominator of (242) by $e^{+jk_{y_2}D}$ reveals

$$A_{2a_c} = \frac{j\ k_r}{4\pi\ k_{y_2}} R_{21} \left[R_{23}\, e^{+jk_{y_2} y_o}\, e^{-j2k_{y_2} D} + e^{-jk_{y_2} y_o} \right] /$$

$$\left[1 - R_{21} R_{23}\, e^{-j2k_{y_2} D} \right]. \tag{243}$$

Factoring $e^{-jk_{y_2} y_o}$ out of the numerator and rearranging yields

$$A_{2a_c} = \frac{j\ k_r}{4\pi\ k_{y_2}} R_{21} \left[1 + R_{23}\, e^{+j2k_{y_2} y_o}\, e^{-j2k_{y_2} D} \right] e^{-jk_{y_2} y_o} /$$

$$\left[1 - R_{21} R_{23}\, e^{-j2k_{y_2} D} \right]. \tag{244}$$

Rearranging (244) provides us with the following desired expression:

$$A_{2a_c} = \frac{jR_{21}}{4\pi} \frac{\left[1 + R_{23}\, e^{-j2k_{y_2}(D - y_0)}\right] e^{-jk_{y_2}y_0}}{\left[1 - R_{21}\, R_{23}\, e^{-j2k_{y_2}D}\right] k_{y_2}} k_r\,. \tag{245}$$

This is exactly the equation derived by Ziomek (1991, equation (3.9-50)) for this unknown constant for the classical waveguide case.

For the unknown constant B_{2a}, *Mathematica* provided the following numerator:

$$\begin{aligned} num_3 = G_1 \big[& a_{1,2}\, a_{2,5}\, a_{3,6}\, a_{4,1}\, a_{5,4} - a_{1,1}\, a_{2,5}\, a_{3,6}\, a_{4,2}\, a_{5,4} \\ & - a_{1,2}\, a_{2,4}\, a_{3,6}\, a_{4,1}\, a_{5,5} + a_{1,1}\, a_{2,4}\, a_{3,6}\, a_{4,2}\, a_{5,5} \\ & - a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,1}\, a_{5,6} + a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,1}\, a_{5,6} \\ & + a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,2}\, a_{5,6} - a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,2}\, a_{5,6} \big]\,. \end{aligned} \tag{246}$$

Factoring (246) and collecting common terms yields the following *generic expression* for the *numerator* of B_{2a}:

$$\begin{aligned} num_3 = G_1 \big[& (a_{1,1}\, a_{2,5}\, a_{4,2} - a_{1,2}\, a_{2,5}\, a_{4,1})(a_{3,4}\, a_{5,6} - a_{3,6}\, a_{5,4}) \\ & + (a_{1,1}\, a_{2,4}\, a_{4,2} - a_{1,2}\, a_{2,4}\, a_{4,1})(a_{3,6}\, a_{5,5} - a_{3,5}\, a_{5,6}) \big]\,. \end{aligned} \tag{247}$$

Substituting appropriate expressions into (247) and using (228) yields the following *general expression* for B_{2a}:

$$B_{2a} = \frac{-k_r}{2\pi}\left(\rho_1(0)\, Y_1^-(0)\, \frac{dY_{2a}^+(0)}{dy} - \rho_2(0)\, \frac{dY_1^-(0)}{dy}\, Y_{2a}^+(0)\right)$$

$$\times \left[Y_{2b}^-(y_o)\left\{\rho_3(D)\, \frac{dY_{2b}^+(D)}{dy}\, Y_3^+(D) - \rho_2(D)\, Y_{2b}^+(D)\, \frac{dY_3^+(D)}{dy}\right\}\right.$$

$$\left. + Y_{2b}^+(y_o)\left\{\rho_2(D)\, Y_{2b}^-(D)\, \frac{dY_3^+(D)}{dy} - \rho_3(D)\, \frac{dY_{2b}^-(D)}{dy}\, Y_3^+(D)\right\}\right] /$$

$$\left[\rho_1(0)\, Y_1^-(0)\left(\frac{dY_{2a}^+(y_o)}{dy}\frac{dY_{2a}^-(0)}{dy} - \frac{dY_{2a}^+(0)}{dy}\frac{dY_{2a}^-(y_o)}{dy}\right)\right.$$

$$\left. - \rho_2(0)\, \frac{dY_1^-(0)}{dy}\left(\frac{dY_{2a}^+(y_o)}{dy}\, Y_{2a}^-(0) - Y_{2a}^+(0)\, \frac{dY_{2a}^-(y_o)}{dy}\right)\right]$$

$$\times \left[Y_{2b}^-(y_o)\left\{\rho_2(D)\, Y_{2b}^+(D)\, \frac{dY_3^+(D)}{dy} - \rho_3(D)\, \frac{dY_{2b}^+(D)}{dy}\, Y_3^+(D)\right\}\right.$$

$$\left. - Y_{2b}^+(y_o)\left\{\rho_2(D)\, Y_{2b}^-(D)\, \frac{dY_3^+(D)}{dy} - \rho_3(D)\, \frac{dY_{2b}^-(D)}{dy}\, Y_3^+(D)\right\}\right]$$

$$- \left[\rho_1(0)\, Y_1^-(0)\left(Y_{2a}^+(y_o)\, \frac{dY_{2a}^-(0)}{dy} - \frac{dY_{2a}^+(0)}{dy}\, Y_{2a}^-(y_o)\right)\right.$$

$$- \rho_2(0) \frac{dY_1^-(0)}{dy} \left(Y_{2a}^+(y_o)\, Y_{2a}^-(0) - Y_{2a}^+(0)\, Y_{2a}^-(y_o)\right)\Big]$$

$$\times \Big[\frac{dY_{2b}^-(y_o)}{dy} \left\{\rho_2(D)\, Y_{2b}^+(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^+(D)}{dy} Y_3^+(D)\right\}$$

$$- \frac{dY_{2b}^+(y_o)}{dy} \left\{\rho_2(D)\, Y_{2b}^-(D) \frac{dY_3^+(D)}{dy} - \rho_3(D) \frac{dY_{2b}^-(D)}{dy} Y_3^+(D)\right\}\Big] . \qquad (248)$$

For constant speed of sound and ambient density, (248) becomes

$$B_{2a_c} = \frac{k_r}{2\pi} (\rho_1 k_{y_2} + \rho_2 k_{y_1})\, e^{-jk_{y_3} D}$$

$$\times \Big[(\rho_3 k_{y_2} - \rho_2 k_{y_3})\, e^{+jk_{y_2} y_o}\, e^{-jk_{y_2} D} + (\rho_2 k_{y_3} + \rho_3 k_{y_2})\, e^{-jk_{y_2} y_o}\, e^{+jk_{y_2} D} \Big] \Big/$$

$$- j\, 2\, k_{y_2}\, e^{-jk_{y_3} D} \Big\{(\rho_1 k_{y_2} - \rho_2 k_{y_1})(\rho_2 k_{y_3} - \rho_3 k_{y_2})\, e^{-jk_{y_2} D}$$

$$+ (\rho_1 k_{y_2} + \rho_2 k_{y_1})(\rho_2 k_{y_3} + \rho_3 k_{y_2})\, e^{+jk_{y_2} D} \Big\} . \qquad (249)$$

Using the definitions of R_{21} and R_{23} presented earlier, (249) may be reduced to

$$B_{2a_c} = \frac{j}{4\pi} \frac{\left[1 + R_{23}\, e^{-j2k_{y_2}(D - y_0)}\right] e^{-jk_{y_2} y_0}}{\left[1 - R_{21}\, R_{23}\, e^{-j2k_{y_2} D}\right] \quad k_{y_2}} k_r . \tag{250}$$

Multiplying (250) by $\frac{R_{21}}{R_{21}}$, B_{2a_c} may be written as

$$B_{2a_c} = \frac{1}{R_{21}} A_{2a_c} . \tag{251}$$

This is exactly the equation derived by Ziomek (1991, equation (3.9-51)) for this unknown constant for the classical waveguide case.

For the unknown constant A_{2b}, *Mathematica* provided the following numerator:

$$\begin{aligned} num_4 = G_1 \Big[& - a_{1,3}\, a_{2,2}\, a_{3,6}\, a_{4,1}\, a_{5,5} + a_{1,2}\, a_{2,3}\, a_{3,6}\, a_{4,1}\, a_{5,5} \\ & - a_{1,1}\, a_{2,3}\, a_{3,6}\, a_{4,2}\, a_{5,5} + a_{1,1}\, a_{2,2}\, a_{3,6}\, a_{4,3}\, a_{5,5} \\ & + a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,1}\, a_{5,6} - a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,1}\, a_{5,6} \\ & + a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,2}\, a_{5,6} - a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,3}\, a_{5,6} \Big] . \end{aligned} \tag{252}$$

Factoring (252) and collecting common terms yields the following *generic expression* for the *numerator* of A_{2b}:

$$\mathrm{num}_4 = G_1 \left[a_{1,3}\, a_{2,2}\, a_{4,1} - a_{1,2}\, a_{2,3}\, a_{4,1} + a_{1,1}\, a_{2,3}\, a_{4,2} - a_{1,1}\, a_{2,2}\, a_{4,3}\right] (a_{3,5}\, a_{5,6} - a_{3,6}\, a_{5,5})\,. \tag{253}$$

Substituting appropriate expressions into (253) and using (228) yields the following *general expression* for A_{2b}:

$$A_{2b} = \frac{-k_r}{2\pi} \Big[Y_{2a}^-(y_o) \Big(\rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(0) - \rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^+(0)}{dy} \Big)$$

$$- Y_{2a}^+(y_o) \Big(\rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^-(0) - \rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^-(0)}{dy} \Big) \Big]$$

$$\times \Big\{ \rho_3(D) \frac{dY_{2b}^-(D)}{dy} Y_3^+(D) - \rho_2(D)\, Y_{2b}^-(D) \frac{dY_3^+(D)}{dy} \Big\} \Big/$$

$$\Big[\rho_1(0)\, Y_1^-(0) \Big(\frac{dY_{2a}^+(y_o)}{dy} \frac{dY_{2a}^-(0)}{dy} - \frac{dY_{2a}^+(0)}{dy} \frac{dY_{2a}^-(y_o)}{dy} \Big)$$

$$- \rho_2(0) \frac{dY_1^-(0)}{dy} \Big(\frac{dY_{2a}^+(y_o)}{dy} Y_{2a}^-(0) - Y_{2a}^+(0) \frac{dY_{2a}^-(y_o)}{dy} \Big) \Big]$$

$$\times \left[Y_{2b}^{-}(y_0)\left\{\rho_2(D)\,Y_{2b}^{+}(D)\,\frac{dY_3^{+}(D)}{dy} - \rho_3(D)\,\frac{dY_{2b}^{+}(D)}{dy}\,Y_3^{+}(D)\right\}\right.$$

$$\left. - Y_{2b}^{+}(y_0)\left\{\rho_2(D)\,Y_{2b}^{-}(D)\,\frac{dY_3^{+}(D)}{dy} - \rho_3(D)\,\frac{dY_{2b}^{-}(D)}{dy}\,Y_3^{+}(D)\right\}\right]$$

$$- \left[\rho_1(0)\,Y_1^{-}(0)\left(Y_{2a}^{+}(y_0)\,\frac{dY_{2a}^{-}(0)}{dy} - \frac{dY_{2a}^{+}(0)}{dy}\,Y_{2a}^{-}(y_0)\right)\right.$$

$$\left. - \rho_2(0)\,\frac{dY_1^{-}(0)}{dy}\left(Y_{2a}^{+}(y_0)\,Y_{2a}^{-}(0) - Y_{2a}^{+}(0)\,Y_{2a}^{-}(y_0)\right)\right]$$

$$\times \left[\frac{dY_{2b}^{-}(y_0)}{dy}\left\{\rho_2(D)\,Y_{2b}^{+}(D)\,\frac{dY_3^{+}(D)}{dy} - \rho_3(D)\,\frac{dY_{2b}^{+}(D)}{dy}\,Y_3^{+}(D)\right\}\right.$$

$$\left. - \frac{dY_{2b}^{+}(y_0)}{dy}\left\{\rho_2(D)\,Y_{2b}^{-}(D)\,\frac{dY_3^{+}(D)}{dy} - \rho_3(D)\,\frac{dY_{2b}^{-}(D)}{dy}\,Y_3^{+}(D)\right\}\right]. \qquad (254)$$

For constant speed of sound and ambient density, (254) becomes

$$A_{2b_c} = \frac{k_r}{2\pi}\, e^{-jk_{y_3}D}\left[\left(\rho_1\,k_{y_2} + \rho_2\,k_{y_1}\right) e^{+jk_{y_2}y_0}\, e^{+jk_{y_2}D}\right.$$

$$+ (\rho_1 k_{y_2} - \rho_2 k_{y_1}) e^{-jk_{y_2} y_o} e^{+jk_{y_2} D}] (\rho_2 k_{y_3} + \rho_3 k_{y_2}) /$$

$$\Big[- j\, 2\, k_{y_2} e^{-jk_{y_3} D} \Big\{ (\rho_1 k_{y_2} - \rho_2 k_{y_1}) (\rho_2 k_{y_3} - \rho_3 k_{y_2}) e^{-jk_{y_2} D}$$

$$+ (\rho_1 k_{y_2} + \rho_2 k_{y_1}) (\rho_2 k_{y_3} + \rho_3 k_{y_2}) e^{+jk_{y_2} D} \Big\} \Big] . \qquad (255)$$

Using the definitions of R_{21} and R_{23} presented earlier, (255) may be reduced to

$$A_{2b_c} = \frac{j}{4\pi} \frac{\left[R_{21} e^{-jk_{y_2} y_o} + e^{+jk_{y_2} y_o} \right]}{\left[1 - R_{21} R_{23} e^{-j2k_{y_2} D} \right]} \frac{k_r}{k_{y_2}} . \qquad (256)$$

This is exactly the equation derived by Ziomek (1991, equation (3.9-52)) for this unknown constant for the classical waveguide case.

For the unknown constant B_{2b}, *Mathematica* provided the following numerator:

$$
\begin{aligned}
num_5 = G_1 \big[& a_{1,3}\, a_{2,2}\, a_{3,6}\, a_{4,1}\, a_{5,4} - a_{1,2}\, a_{2,3}\, a_{3,6}\, a_{4,1}\, a_{5,4} \\
& + a_{1,1}\, a_{2,3}\, a_{3,6}\, a_{4,2}\, a_{5,4} - a_{1,1}\, a_{2,2}\, a_{3,6}\, a_{4,3}\, a_{5,4} \\
& - a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,1}\, a_{5,6} + a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,1}\, a_{5,6} \\
& - a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,2}\, a_{5,6} + a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,3}\, a_{5,6} \big] .
\end{aligned}
\tag{257}
$$

Factoring (257) and collecting common terms yields the following *generic expression* for the *numerator* of B_{2b}:

$$
\begin{aligned}
num_5 = G_1 \big[& a_{1,3}\, a_{2,2}\, a_{4,1} - a_{1,2}\, a_{2,3}\, a_{4,1} + a_{1,1}\, a_{2,3}\, a_{4,2} - \\
& a_{1,1}\, a_{2,2}\, a_{4,3} \big] \left(a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6} \right) .
\end{aligned}
\tag{258}
$$

Substituting appropriate expressions into (258) and using (228) yields the following *general expression* for B_{2b}:

$$
\begin{aligned}
B_{2b} = \frac{-k_f}{2\pi} \Big[& Y_{2a}^{-}(y_o) \left(\rho_2(0) \frac{dY_1^{-}(0)}{dy} Y_{2a}^{+}(0) - \rho_1(0)\, Y_1^{-}(0) \frac{dY_{2a}^{+}(0)}{dy} \right) \\
& - Y_{2a}^{+}(y_o) \left(\rho_2(0) \frac{dY_1^{-}(0)}{dy} Y_{2a}^{-}(0) - \rho_1(0)\, Y_1^{-}(0) \frac{dY_{2a}^{-}(0)}{dy} \right) \Big] \\
& \times \left\{ \rho_2(D)\, Y_{2b}^{+}(D) \frac{dY_3^{+}(D)}{dy} - \rho_3(D) \frac{dY_{2b}^{+}(D)}{dy} Y_3^{+}(D) \right\} /
\end{aligned}
$$

$$\Big[\rho_1(0)\, Y_1^-(0) \Big(\frac{dY_{2a}^+(y_0)}{dy}\frac{dY_{2a}^-(0)}{dy} - \frac{dY_{2a}^+(0)}{dy}\frac{dY_{2a}^-(y_0)}{dy}\Big)$$

$$- \rho_2(0)\frac{dY_1^-(0)}{dy}\Big(\frac{dY_{2a}^+(y_0)}{dy} Y_{2a}^-(0) - Y_{2a}^+(0)\frac{dY_{2a}^-(y_0)}{dy}\Big)\Big]$$

$$\times \Big[Y_{2b}^-(y_0)\Big\{\rho_2(D)\, Y_{2b}^+(D)\frac{dY_3^+(D)}{dy} - \rho_3(D)\frac{dY_{2b}^+(D)}{dy} Y_3^+(D)\Big\}$$

$$- Y_{2b}^+(y_0)\Big\{\rho_2(D)\, Y_{2b}^-(D)\frac{dY_3^+(D)}{dy} - \rho_3(D)\frac{dY_{2b}^-(D)}{dy} Y_3^+(D)\Big\}\Big]$$

$$- \Big[\rho_1(0)\, Y_1^-(0)\Big(Y_{2a}^+(y_0)\frac{dY_{2a}^-(0)}{dy} - \frac{dY_{2a}^+(0)}{dy} Y_{2a}^-(y_0)\Big)$$

$$- \rho_2(0)\frac{dY_1^-(0)}{dy}\Big(Y_{2a}^+(y_0)\, Y_{2a}^-(0) - Y_{2a}^+(0)\, Y_{2a}^-(y_0)\Big)\Big]$$

$$\times \Big[\frac{dY_{2b}^-(y_0)}{dy}\Big\{\rho_2(D)\, Y_{2b}^+(D)\frac{dY_3^+(D)}{dy} - \rho_3(D)\frac{dY_{2b}^+(D)}{dy} Y_3^+(D)\Big\}$$

$$- \frac{dY_{2b}^+(y_0)}{dy}\Big\{\rho_2(D)\, Y_{2b}^-(D)\frac{dY_3^+(D)}{dy} - \rho_3(D)\frac{dY_{2b}^-(D)}{dy} Y_3^+(D)\Big\}\Big]\,. \qquad (259)$$

For constant speed of sound and ambient density, (259) becomes

$$B_{2b_c} = \frac{k_r}{2\pi} e^{-jk_{y_2} D} \; e^{-jk_{y_3} D} \Big[(\rho_1 k_{y_2} + \rho_2 k_{y_1}) e^{+jk_{y_2} y_o}$$

$$+ (\rho_1 k_{y_2} - \rho_2 k_{y_1}) e^{-jk_{y_2} y_o} \Big] (\rho_3 k_{y_2} - \rho_2 k_{y_3}) \Big/$$

$$\Big[-j\, 2\, k_{y_2} \; e^{-jk_{y_3} D} \Big\{ (\rho_1 k_{y_2} - \rho_2 k_{y_1}) (\rho_2 k_{y_3} - \rho_3 k_{y_2}) e^{-jk_{y_2} D}$$

$$+ (\rho_1 k_{y_2} + \rho_2 k_{y_1}) (\rho_2 k_{y_3} + \rho_3 k_{y_2}) e^{+jk_{y_2} D} \Big\} \Big] . \qquad (260)$$

Using the definitions of R_{21} and R_{23} presented earlier, (260) may be reduced to

$$B_{2b_c} = \frac{j}{4\pi} \frac{\left[R_{21} \; e^{-jk_{y_2} y_o} + e^{+jk_{y_2} y_o} \right]}{\left[1 - R_{21} \; R_{23} \; e^{-j2k_{y_2} D} \right] k_{y_2}} \, k_r \, R_{23} \, e^{-j2k_{y_2} D} , \qquad (261)$$

or, rewriting,

$$B_{2b_c} = A_{2b_c} R_{23}\, e^{-j2k_{y_2} D} \,. \tag{262}$$

This is exactly the equation derived by Ziomek (1991, equation (3.9-53)) for this unknown constant for the classical waveguide case.

For the unknown constant B_1, *Mathematica* provided the following numerator:

$$\begin{aligned} num_1 = G_1 \big[& a_{1,3}\, a_{2,5}\, a_{3,6}\, a_{4,2}\, a_{5,4} - a_{1,2}\, a_{2,5}\, a_{3,6}\, a_{4,3}\, a_{5,4} \\ & - a_{1,3}\, a_{2,4}\, a_{3,6}\, a_{4,2}\, a_{5,5} + a_{1,2}\, a_{2,4}\, a_{3,6}\, a_{4,3}\, a_{5,5} \\ & - a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,2}\, a_{5,6} + a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,2}\, a_{5,6} \\ & + a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,3}\, a_{5,6} - a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,3}\, a_{5,6} \big] \,. \end{aligned} \tag{263}$$

Factoring (263) and collecting common terms yields the following *generic expression* for the *numerator* of B_1:

$$\begin{aligned} num_1 = G_1 \Big[& \big(a_{1,3}\, a_{2,5}\, a_{4,2} - a_{1,2}\, a_{2,5}\, a_{4,3}\big)\big(a_{3,6}\, a_{5,4} - a_{3,4}\, a_{5,6}\big) \\ & + \big(a_{1,3}\, a_{2,4}\, a_{4,2} - a_{1,2}\, a_{2,4}\, a_{4,3}\big)\big(a_{3,5}\, a_{5,6} - a_{3,5}\, a_{5,5}\big) \Big] \,. \end{aligned} \tag{264}$$

Substituting appropriate expressions into (264) and using (228) yields the following *general expression* for B_1:

$$B_1 = \frac{-k_r}{2\pi}\, \rho_2(0) \left(Y^{+}_{2a}(0) \frac{dY^{-}_{2a}(0)}{dy} - \frac{dY^{+}_{2a}(0)}{dy} Y^{-}_{2a}(0) \right)$$

$$\times \left[Y^{-}_{2b}(y_o)\left\{\rho_2(D)\,Y^{+}_{2b}(D)\,\frac{dY^{+}_{3}(D)}{dy} - \rho_3(D)\,\frac{dY^{+}_{2b}(D)}{dy}\,Y^{+}_{3}(D)\right\}\right.$$

$$\left. + Y^{+}_{2b}(y_o)\left\{\rho_3(D)\,\frac{dY^{-}_{2b}(D)}{dy}\,Y^{+}_{3}(D) - \rho_2(D)\,Y^{-}_{2b}(D)\,\frac{dY^{+}_{3}(D)}{dy}\right\}\right] /$$

$$\left[\rho_1(0)\,Y^{-}_{1}(0)\left(\frac{dY^{+}_{2a}(y_o)}{dy}\,\frac{dY^{-}_{2a}(0)}{dy} - \frac{dY^{+}_{2a}(0)}{dy}\,\frac{dY^{-}_{2a}(y_o)}{dy}\right)\right.$$

$$\left. - \rho_2(0)\,\frac{dY^{-}_{1}(0)}{dy}\left(\frac{dY^{+}_{2a}(y_o)}{dy}\,Y^{-}_{2a}(0) - Y^{+}_{2a}(0)\,\frac{dY^{-}_{2a}(y_o)}{dy}\right)\right]$$

$$\times \left[Y^{-}_{2b}(y_o)\left\{\rho_2(D)\,Y^{+}_{2b}(D)\,\frac{dY^{+}_{3}(D)}{dy} - \rho_3(D)\,\frac{dY^{+}_{2b}(D)}{dy}\,Y^{+}_{3}(D)\right\}\right.$$

$$\left. - Y^{+}_{2b}(y_o)\left\{\rho_2(D)\,Y^{-}_{2b}(D)\,\frac{dY^{+}_{3}(D)}{dy} - \rho_3(D)\,\frac{dY^{-}_{2b}(D)}{dy}\,Y^{+}_{3}(D)\right\}\right]$$

$$- \left[\rho_1(0)\,Y^{-}_{1}(0)\left(Y^{+}_{2a}(y_o)\,\frac{dY^{-}_{2a}(0)}{dy} - \frac{dY^{+}_{2a}(0)}{dy}\,Y^{-}_{2a}(y_o)\right)\right.$$

$$\left. - \rho_2(0)\,\frac{dY^{-}_{1}(0)}{dy}\left(Y^{+}_{2a}(y_o)\,Y^{-}_{2a}(0) - Y^{+}_{2a}(0)\,Y^{-}_{2a}(y_o)\right)\right]$$

$$\times \left[\frac{dY^-_{2b}(y_o)}{dy} \left\{ \rho_2(D)\, Y^+_{2b}(D) \frac{dY^+_3(D)}{dy} - \rho_3(D) \frac{dY^+_{2b}(D)}{dy} Y^+_3(D) \right\} \right.$$

$$\left. - \frac{dY^+_{2b}(y_o)}{dy} \left\{ \rho_2(D)\, Y^-_{2b}(D) \frac{dY^+_3(D)}{dy} - \rho_3(D) \frac{dY^-_{2b}(D)}{dy} Y^+_3(D) \right\} \right] . \qquad (265)$$

For constant speed of sound and ambient density, (265) becomes

$$B_{1_c} = \frac{k_r}{2\pi} (2\, \rho_2\, k_{y_2})\, e^{-jk_{y_3} D} \left[(\rho_3\, k_{y_2} - \rho_2\, k_{y_3})\, e^{+jk_{y_2} y_o}\, e^{-jk_{y_2} D} \right.$$

$$\left. + (\rho_3\, k_{y_2} + \rho_2\, k_{y_3})\, e^{-jk_{y_2} y_o}\, e^{+k_{y_2} D} \right] /$$

$$\left[-j\, 2\, k_{y_2}\, e^{-jk_{y_3} D} \left\{ (\rho_1\, k_{y_2} - \rho_2\, k_{y_1})(\rho_2\, k_{y_3} - \rho_3\, k_{y_2})\, e^{-jk_{y_2} D} \right. \right.$$

$$\left. \left. + (\rho_1\, k_{y_2} + \rho_2\, k_{y_1})(\rho_2\, k_{y_3} + \rho_3\, k_{y_2})\, e^{+jk_{y_2} D} \right\} \right] . \qquad (266)$$

Using the definitions of R_{21} and R_{23} presented earlier, and defining the transmission coefficient at the boundary between medium two and medium one T_{21} (see Ziomek (1991, equation (3.9-58))) as follows:

$$T_{21} = \frac{2\ \rho_2\ k_{y_2}}{\rho_1\ k_{y_2} + \rho_2\ k_{y_1}}, \tag{267}$$

(266) may be reduced to

$$B_{1_c} = \frac{j}{4\pi} \frac{\left[1 + R_{23}\ e^{+j2k_{y_2} y_o}\ e^{-j2k_{y_2} D}\right] e^{-jk_{y_2} y_o}}{\left[1 - R_{21}\ R_{23}\ e^{-j2k_{y_2} D}\right] \quad k_{y_2}}\ k_r\, T_{21}\,. \tag{268}$$

Multiplying (268) by $\frac{R_{21}}{R_{21}}$ *allows us to write* B_{1_c} *in the form*

$$B_{1_c} = B_{2a_c} T_{21}\,. \tag{269}$$

This is exactly the equation derived by Ziomek (1991, equation (3.9-56)) for this unknown constant for the classical waveguide case.

Finally, for the unknown constant A_3, *Mathematica* provided the following numerator:

$$\begin{aligned} num_6 = G_1 [& - a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,1}\, a_{5,4} + a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,1}\, a_{5,4} \\ & - a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,2}\, a_{5,4} + a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,3}\, a_{5,4} \end{aligned}$$

$$+ a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,1}\, a_{5,5} - a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,1}\, a_{5,5}$$
$$+ a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,2}\, a_{5,5} - a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,3}\, a_{5,5}] \,. \qquad (270)$$

Factoring (270) and collecting common terms yields the following *generic expression* for the *numerator* of A_3:

$$num_6 = G_1 \, [(a_{1,3}\, a_{2,2}\, a_{4,1} - a_{1,2}\, a_{2,3}\, a_{4,1} + a_{1,1}\, a_{2,3}\, a_{4,2} - a_{1,1}\, a_{2,2}\, a_{4,3})]\, (a_{3,4}\, a_{5,5} - a_{3,5}\, a_{5,4}) \,. \qquad (271)$$

Substituting appropriate expressions into (271) and using (228) yields the following *general expression* for A_3:

$$A_3 = \frac{-k_r}{2\pi} \rho_2(D) \left(Y^+_{2b}(D) \frac{dY^-_{2b}(D)}{dy} - \frac{dY^+_{2b}(D)}{dy} Y^-_{2b}(D) \right)$$

$$\times \left[Y^-_{2a}(y_o) \left\{ \rho_2(0) \frac{dY^-_1(0)}{dy} Y^+_{2a}(0) - \rho_1(0)\, Y^-_1(0) \frac{dY^+_{2a}(0)}{dy} \right\} \right.$$

$$\left. - Y^+_{2a}(y_o) \left\{ \rho_2(0) \frac{dY^-_1(0)}{dy} Y^-_{2a}(0) - \rho_1(0)\, Y^-_1(0) \frac{dY^-_{2a}(0)}{dy} \right\} \right] \Big/$$

$$\left[\rho_1(0)\, Y^-_1(0) \left(\frac{dY^+_{2a}(y_o)}{dy} \frac{dY^-_{2a}(0)}{dy} - \frac{dY^+_{2a}(0)}{dy} \frac{dY^-_{2a}(y_o)}{dy} \right) \right.$$

$$\left. - \rho_2(0) \frac{dY^-_1(0)}{dy} \left(\frac{dY^+_{2a}(y_o)}{dy} Y^-_{2a}(0) - Y^+_{2a}(0) \frac{dY^-_{2a}(y_o)}{dy} \right) \right]$$

$$\times \left[Y_{2b}^{-}(y_o)\left\{\rho_2(D)\,Y_{2b}^{+}(D)\frac{dY_3^{+}(D)}{dy} - \rho_3(D)\frac{dY_{2b}^{+}(D)}{dy}Y_3^{+}(D)\right\}\right.$$

$$\left. - Y_{2b}^{+}(y_o)\left\{\rho_2(D)\,Y_{2b}^{-}(D)\frac{dY_3^{+}(D)}{dy} - \rho_3(D)\frac{dY_{2b}^{-}(D)}{dy}Y_3^{+}(D)\right\}\right]$$

$$- \left[\rho_1(0)\,Y_1^{-}(0)\left(Y_{2a}^{+}(y_o)\frac{dY_{2a}^{-}(0)}{dy} - \frac{dY_{2a}^{+}(0)}{dy}Y_{2a}^{-}(y_o)\right)\right.$$

$$\left. - \rho_2(0)\frac{dY_1^{-}(0)}{dy}\left(Y_{2a}^{+}(y_o)\,Y_{2a}^{-}(0) - Y_{2a}^{+}(0)\,Y_{2a}^{-}(y_o)\right)\right]$$

$$\times \left[\frac{dY_{2b}^{-}(y_o)}{dy}\left\{\rho_2(D)\,Y_{2b}^{+}(D)\frac{dY_3^{+}(D)}{dy} - \rho_3(D)\frac{dY_{2b}^{+}(D)}{dy}Y_3^{+}(D)\right\}\right.$$

$$\left. - \frac{dY_{2b}^{+}(y_o)}{dy}\left\{\rho_2(D)\,Y_{2b}^{-}(D)\frac{dY_3^{+}(D)}{dy} - \rho_3(D)\frac{dY_{2b}^{-}(D)}{dy}Y_3^{+}(D)\right\}\right]. \qquad (272)$$

For constant speed of sound and ambient density, (272) becomes

$$A_{3_c} = \frac{k_r}{2\pi}(2\,\rho_2\,k_{y_2})\left[(\rho_1\,k_{y_2} + \rho_2\,k_{y_1})\,e^{+jk_{y_2}y_o}\right.$$

$$+ (\rho_1 k_{y_2} - \rho_2 k_{y_1}) e^{-jk_{y_2} y_0} \Big] \Big/$$

$$\Big[- j\, 2\, k_{y_2}\, e^{-jk_{y_3} D} \Big\{ (\rho_1 k_{y_2} - \rho_2 k_{y_1})(\rho_2 k_{y_3} - \rho_3 k_{y_2}) e^{-jk_{y_2} D}$$

$$+ (\rho_1 k_{y_2} + \rho_2 k_{y_1})(\rho_2 k_{y_3} + \rho_3 k_{y_2}) e^{+jk_{y_2} D} \Big\} \Big] . \tag{273}$$

Using the definitions of R_{21} and R_{23} presented earlier, and defining the transmission coefficient at the boundary between medium two and medium three T_{23} (see Ziomek (1991, equation (3.9-59))) as follows:

$$T_{23} = \frac{2\ \rho_2\ k_{y_2}}{\rho_3\ k_{y_2}\ +\ \rho_2\ k_{y_3}} , \tag{274}$$

(273) may be reduced to

$$A_{3_c} = \frac{j}{4\pi} \frac{\left[R_{21}\, e^{-jk_{y_2} y_o} + e^{+jk_{y_2} y_o}\right]}{\left[1 - R_{21} R_{23}\, e^{-j2k_{y_2} D}\right]} \frac{k_r}{k_{y_2}} \frac{e^{-jk_{y_2} D}}{e^{-jk_{y_3} D}} T_{23} . \tag{275}$$

(275) may be rewritten as

$$A_{3_c} = A_{2b_c} T_{23} \frac{e^{-jk_{y_2} D}}{e^{-jk_{y_3} D}} . \tag{276}$$

This is exactly the equation derived by Ziomek (1991, equation (3.9-57)) for this unknown constant for the classical waveguide case.

To summarize the results for the three media waveguide with plane, parallel boundaries, *generic* expressions for the numerators of the unknown constants are given by (232), (247), (253), (258), (264), and (271). The *generic* expression for the common denominator is given by (226). The fact that these constants have a common denominator may be useful when constructing a modular program for numerical calculations involving these constants. For the *general* case, in which speed of sound and density are

arbitrary functions of depth, the unknown constants are given by (235), (248), (254), (259), (265), and (272). Again, it should be noted that the denominators in these expressions are exactly the same, offering the same advantages in modular programming. For constant speed of sound and density, the unknown constants are given by (245), (251), (256), (262), (269), and (276).

The fact that *Mathematica*'s output can be shown to be equivalent to results derived in a completely independent manner gives us some confidence in the answers arrived at by *Mathematica*'s LinearSolve function and symbolic processing. We will now program *Mathematica* to solve the four media waveguide problem assuming all boundaries are planar and parallel. By obtaining constant sound speed, constant density expressions for the resulting eight unknown constants and making judicious selections of transmission and reflection coefficients, we can mathematically "remove" the fourth medium and verify that the four media case correctly reduces to the three media case. This will offer us more confidence in the program's output, and provide some meaningful results.

For the four media waveguide problem, the vector-matrix quantities of (198) are defined as follows:

$$
\mathbf{A} = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & 0 & 0 & 0 & 0 & 0 \\
0 & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & 0 & 0 & 0 \\
0 & 0 & 0 & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{4,6} & a_{4,7} & a_{4,8} \\
a_{5,1} & a_{5,2} & a_{5,3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{7,6} & a_{7,7} & a_{7,8} \\
0 & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & 0 & 0 & 0
\end{bmatrix}
$$

where

$$a_{1,1} = -\rho_1(y_S)\, Y_1^-(y_S) \tag{277}$$

$$a_{1,2} = \rho_2(y_S)\, Y_{2a}^+(y_S) \tag{278}$$

$$a_{1,3} = \rho_2(y_S)\, Y_{2a}^-(y_S) \tag{279}$$

$$a_{2,2} = Y_{2a}^+(y_o) \tag{280}$$

$$a_{2,3} = Y_{2a}^-(y_o) \tag{281}$$

$$a_{2,4} = -Y_{2b}^+(y_o) \tag{282}$$

$$a_{2,5} = -Y_{2b}^-(y_o) \tag{283}$$

$$a_{3,4} = \rho_2(y_{B_1})\, Y_{2b}^+(y_{B_1}) \tag{284}$$

$$a_{3,5} = \rho_2(y_{B_1})\, Y_{2b}^-(y_{B_1}) \tag{285}$$

$$a_{3,6} = -\rho_3(y_{B_1})\, Y_3^+(y_{B_1}) \tag{286}$$

$$a_{3,7} = -\rho_3(y_{B_1})\, Y_3^-(y_{B_1}) \tag{287}$$

$$a_{4,6} = \rho_3(y_{B_2})\, Y_3^+(y_{B_2}) \tag{288}$$

$$a_{4,7} = \rho_3(y_{B_2})\, Y_3^-(y_{B_2}) \tag{289}$$

$$a_{4,8} = -\rho_4(y_{B_2})\, Y_4^+(y_{B_2}) \tag{290}$$

$$a_{5,1} = -\frac{dY_1^-(y_S)}{dy} \tag{291}$$

$$a_{5,2} = \frac{dY_{2a}^+(y_S)}{dy} \tag{292}$$

$$a_{5,3} = \frac{dY_{2a}^-(y_S)}{dy} \tag{293}$$

$$a_{6,4} = \frac{dY_{2b}^+(y_{B_1})}{dy} \tag{294}$$

$$a_{6,5} = \frac{dY_{2b}^-(y_{B_1})}{dy} \tag{295}$$

$$a_{6,6} = -\frac{dY_3^+(y_{B_1})}{dy} \tag{296}$$

$$a_{6,7} = -\frac{dY_3^-(y_{B_1})}{dy} \tag{297}$$

$$a_{7,6} = \frac{dY_3^+(y_{B_2})}{dy} \tag{298}$$

$$a_{7,7} = \frac{dY_3^-(y_{B_2})}{dy} \tag{299}$$

$$a_{7,8} = -\frac{dY_4^{+}(y_{B_2})}{dy} \tag{300}$$

$$a_{8,2} = \frac{dY_{2a}^{+}(y_o)}{dy} \tag{301}$$

$$a_{8,3} = \frac{dY_{2a}^{-}(y_o)}{dy} \tag{302}$$

$$a_{8,4} = -\frac{dY_{2b}^{+}(y_o)}{dy} \tag{303}$$

$$a_{8,5} = -\frac{dY_{2b}^{-}(y_o)}{dy}; \tag{304}$$

$$\mathbf{x} = \begin{bmatrix} B_1 & A_{2a} & B_{2a} & A_{2b} & B_{2b} & A_3 & B_3 & A_4 \end{bmatrix}^T \tag{305}$$

where the superscript T indicates the transpose matrix operator (indicating that $\mathbf{x}$ is a column vector),

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_1 \end{bmatrix}^T \tag{306}$$

where the superscript T indicates that $\mathbf{b}$ is a column vector, and

$$G_1 = -\frac{k_r}{2\pi}. \tag{307}$$

Since the matrix **A** is an eight by eight square matrix for this four media waveguide problem, the solution to the system can be written as described in (222). The *Mathematica* code required to perform this task is as follows:

```
a = {{a1c1, a1c2, a1c3, 0, 0, 0, 0, 0},
     {0, a2c2, a2c3, a2c4, a2c5, 0, 0, 0},
     {0, 0, 0, a3c4, a3c5, a3c6, a3c7, 0},
     {0, 0, 0, 0, 0, a4c6, a4c7, a4c8},
     {a5c1, a5c2, a5c3, 0, 0, 0, 0, 0},
     {0, 0, 0, a6c4, a6c5, a6c6, a6c7, 0},
     {0, 0, 0, 0, 0, a7c6, a7c7, a7c8},
     {0, a8c2, a8c3, a8c4, a8c5, 0, 0, 0}};

b = {0, 0, 0, 0, 0, 0, 0, G1};

x = (Inverse[a]).b
```

This code resulted in halted execution due to a singularity error in the evaluation of the matrix inverse. Luckily, the LinearSolve function can be used to solve (198) directly for this problem because **A** is an eight by eight square matrix. The following modified *Mathematica* code was developed:

```
a = {{a1c1, a1c2, a1c3, 0, 0, 0, 0, 0},
     {0, a2c2, a2c3, a2c4, a2c5, 0, 0, 0},
     {0, 0, 0, a3c4, a3c5, a3c6, a3c7, 0},
     {0, 0, 0, 0, 0, a4c6, a4c7, a4c8},
     {a5c1, a5c2, a5c3, 0, 0, 0, 0, 0},
     {0, 0, 0, a6c4, a6c5, a6c6, a6c7, 0},
     {0, 0, 0, 0, 0, a7c6, a7c7, a7c8},
     {0, a8c2, a8c3, a8c4, a8c5, 0, 0, 0}};

b = {0, 0, 0, 0, 0, 0, 0, G1};

LinearSolve[a,b]
```

This revised code ran successfully, yielding results in the same format as the three media waveguide problem discussed earlier [see (223)]. Again, when *Mathematica* functions such as Factor, Cancel, and Simplify were

applied to the output, the same result was returned, indicating that the program was satisfied that the output was as simple as it could make it. This fact confirms our suspicions that the symbolic algebra routines contained in *Mathematica* lack sufficient sophistication for this application.

We will now present the results of the program for the four media waveguide problem. The analysis of these results will be conducted in a manner similar to the analysis conducted for the three media waveguide problem. That is, the results will be simplified manually in order to generate generic expressions for the unknown constants in terms of the generic elements. Then, (277) through (304) and (307) will be substituted into the generic expressions to reveal general expressions for these unknown constants. Next, we will assume constant speed of sound and constant ambient density, and develop a set of expressions for the unknown constants. Finally, we will make some judicious assumptions regarding the reflection coefficient (at the boundary between medium three and medium four), the transmission coefficient (at the boundary between medium three and medium four), and the location of the boundary y_{B_2} (thereby mathematically removing the fourth medium) to show that the constant speed of sound, constant density expressions for the four media waveguide problem reduce to the results of the three media waveguide problem already verified. This verification will be conducted in the following order: A_{2a}, B_{2a}, A_{2b}, B_{2b}, A_3, B_3, B_1, and A_4. We will demonstrate this entire process for the unknown constant A_{2a} only, and simply present the results for the other seven unknown constants.

The first output expression to be explored is the *common denominator*.

denom

$$
\begin{aligned}
\text{denom} = &- a_{1,3} a_{2,5} a_{3,7} a_{4,8} a_{5,1} a_{6,4} a_{7,6} a_{8,2} \\
&+ a_{1,1} a_{2,5} a_{3,7} a_{4,8} a_{5,3} a_{6,4} a_{7,6} a_{8,2} + a_{1,3} a_{2,4} a_{3,7} a_{4,8} a_{5,1} a_{6,5} a_{7,6} a_{8,2} \\
&- a_{1,1} a_{2,4} a_{3,7} a_{4,8} a_{5,3} a_{6,5} a_{7,6} a_{8,2} + a_{1,3} a_{2,5} a_{3,4} a_{4,8} a_{5,1} a_{6,7} a_{7,6} a_{8,2} \\
&- a_{1,3} a_{2,4} a_{3,5} a_{4,8} a_{5,1} a_{6,7} a_{7,6} a_{8,2} - a_{1,1} a_{2,5} a_{3,4} a_{4,8} a_{5,3} a_{6,7} a_{7,6} a_{8,2} \\
&+ a_{1,1} a_{2,4} a_{3,5} a_{4,8} a_{5,3} a_{6,7} a_{7,6} a_{8,2} + a_{1,3} a_{2,5} a_{3,6} a_{4,8} a_{5,1} a_{6,4} a_{7,7} a_{8,2} \\
&- a_{1,1} a_{2,5} a_{3,6} a_{4,8} a_{5,3} a_{6,4} a_{7,7} a_{8,2} - a_{1,3} a_{2,4} a_{3,6} a_{4,8} a_{5,1} a_{6,5} a_{7,7} a_{8,2} \\
&+ a_{1,1} a_{2,4} a_{3,6} a_{4,8} a_{5,3} a_{6,5} a_{7,7} a_{8,2} - a_{1,3} a_{2,5} a_{3,4} a_{4,8} a_{5,1} a_{6,6} a_{7,7} a_{8,2} \\
&+ a_{1,3} a_{2,4} a_{3,5} a_{4,8} a_{5,1} a_{6,6} a_{7,7} a_{8,2} + a_{1,1} a_{2,5} a_{3,4} a_{4,8} a_{5,3} a_{6,6} a_{7,7} a_{8,2} \\
&- a_{1,1} a_{2,4} a_{3,5} a_{4,8} a_{5,3} a_{6,6} a_{7,7} a_{8,2} + a_{1,3} a_{2,5} a_{3,7} a_{4,6} a_{5,1} a_{6,4} a_{7,8} a_{8,2} \\
&- a_{1,3} a_{2,5} a_{3,6} a_{4,7} a_{5,1} a_{6,4} a_{7,8} a_{8,2} - a_{1,1} a_{2,5} a_{3,7} a_{4,6} a_{5,3} a_{6,4} a_{7,8} a_{8,2} \\
&+ a_{1,1} a_{2,5} a_{3,6} a_{4,7} a_{5,3} a_{6,4} a_{7,8} a_{8,2} - a_{1,3} a_{2,4} a_{3,7} a_{4,6} a_{5,1} a_{6,5} a_{7,8} a_{8,2} \\
&+ a_{1,3} a_{2,4} a_{3,6} a_{4,7} a_{5,1} a_{6,5} a_{7,8} a_{8,2} + a_{1,1} a_{2,4} a_{3,7} a_{4,6} a_{5,3} a_{6,5} a_{7,8} a_{8,2} \\
&- a_{1,1} a_{2,4} a_{3,6} a_{4,7} a_{5,3} a_{6,5} a_{7,8} a_{8,2} + a_{1,3} a_{2,5} a_{3,4} a_{4,7} a_{5,1} a_{6,6} a_{7,8} a_{8,2} \\
&- a_{1,3} a_{2,4} a_{3,5} a_{4,7} a_{5,1} a_{6,6} a_{7,8} a_{8,2} - a_{1,1} a_{2,5} a_{3,4} a_{4,7} a_{5,3} a_{6,6} a_{7,8} a_{8,2} \\
&+ a_{1,1} a_{2,4} a_{3,5} a_{4,7} a_{5,3} a_{6,6} a_{7,8} a_{8,2} - a_{1,3} a_{2,5} a_{3,4} a_{4,6} a_{5,1} a_{6,7} a_{7,8} a_{8,2} \\
&+ a_{1,3} a_{2,4} a_{3,5} a_{4,6} a_{5,1} a_{6,7} a_{7,8} a_{8,2} + a_{1,1} a_{2,5} a_{3,4} a_{4,6} a_{5,3} a_{6,7} a_{7,8} a_{8,2} \\
&- a_{1,1} a_{2,4} a_{3,5} a_{4,6} a_{5,3} a_{6,7} a_{7,8} a_{8,2} + a_{1,2} a_{2,5} a_{3,7} a_{4,8} a_{5,1} a_{6,4} a_{7,6} a_{8,3} \\
&- a_{1,1} a_{2,5} a_{3,7} a_{4,8} a_{5,2} a_{6,4} a_{7,6} a_{8,3} - a_{1,2} a_{2,4} a_{3,7} a_{4,8} a_{5,1} a_{6,5} a_{7,6} a_{8,3} \\
&+ a_{1,1} a_{2,4} a_{3,7} a_{4,8} a_{5,2} a_{6,5} a_{7,6} a_{8,3} - a_{1,2} a_{2,5} a_{3,4} a_{4,8} a_{5,1} a_{6,7} a_{7,6} a_{8,3} \\
&+ a_{1,2} a_{2,4} a_{3,5} a_{4,8} a_{5,1} a_{6,7} a_{7,6} a_{8,3} + a_{1,1} a_{2,5} a_{3,4} a_{4,8} a_{5,2} a_{6,7} a_{7,6} a_{8,3} \\
&- a_{1,1} a_{2,4} a_{3,5} a_{4,8} a_{5,2} a_{6,7} a_{7,6} a_{8,3} - a_{1,2} a_{2,5} a_{3,6} a_{4,8} a_{5,1} a_{6,4} a_{7,7} a_{8,3} \\
&+ a_{1,1} a_{2,5} a_{3,6} a_{4,8} a_{5,2} a_{6,4} a_{7,7} a_{8,3} + a_{1,2} a_{2,4} a_{3,6} a_{4,8} a_{5,1} a_{6,5} a_{7,7} a_{8,3} \\
&- a_{1,1} a_{2,4} a_{3,6} a_{4,8} a_{5,2} a_{6,5} a_{7,7} a_{8,3} + a_{1,2} a_{2,5} a_{3,4} a_{4,8} a_{5,1} a_{6,6} a_{7,7} a_{8,3} \\
&- a_{1,2} a_{2,4} a_{3,5} a_{4,8} a_{5,1} a_{6,6} a_{7,7} a_{8,3} - a_{1,1} a_{2,5} a_{3,4} a_{4,8} a_{5,2} a_{6,6} a_{7,7} a_{8,3} \\
&+ a_{1,1} a_{2,4} a_{3,5} a_{4,8} a_{5,2} a_{6,6} a_{7,7} a_{8,3} - a_{1,2} a_{2,5} a_{3,7} a_{4,6} a_{5,1} a_{6,4} a_{7,8} a_{8,3} \\
&+ a_{1,2} a_{2,5} a_{3,6} a_{4,7} a_{5,1} a_{6,4} a_{7,8} a_{8,3} + a_{1,1} a_{2,5} a_{3,7} a_{4,6} a_{5,2} a_{6,4} a_{7,8} a_{8,3} \\
&- a_{1,1} a_{2,5} a_{3,6} a_{4,7} a_{5,2} a_{6,4} a_{7,8} a_{8,3} + a_{1,2} a_{2,4} a_{3,7} a_{4,6} a_{5,1} a_{6,5} a_{7,8} a_{8,3} \\
&- a_{1,2} a_{2,4} a_{3,6} a_{4,7} a_{5,1} a_{6,5} a_{7,8} a_{8,3} - a_{1,1} a_{2,4} a_{3,7} a_{4,6} a_{5,2} a_{6,5} a_{7,8} a_{8,3} \\
&+ a_{1,1} a_{2,4} a_{3,6} a_{4,7} a_{5,2} a_{6,5} a_{7,8} a_{8,3} - a_{1,2} a_{2,5} a_{3,4} a_{4,7} a_{5,1} a_{6,6} a_{7,8} a_{8,3} \\
&+ a_{1,2} a_{2,4} a_{3,5} a_{4,7} a_{5,1} a_{6,6} a_{7,8} a_{8,3} + a_{1,1} a_{2,5} a_{3,4} a_{4,7} a_{5,2} a_{6,6} a_{7,8} a_{8,3} \\
&- a_{1,1} a_{2,4} a_{3,5} a_{4,7} a_{5,2} a_{6,6} a_{7,8} a_{8,3} + a_{1,2} a_{2,5} a_{3,4} a_{4,6} a_{5,1} a_{6,7} a_{7,8} a_{8,3} \\
&- a_{1,2} a_{2,4} a_{3,5} a_{4,6} a_{5,1} a_{6,7} a_{7,8} a_{8,3} - a_{1,1} a_{2,5} a_{3,4} a_{4,6} a_{5,2} a_{6,7} a_{7,8} a_{8,3} \\
&+ a_{1,1} a_{2,4} a_{3,5} a_{4,6} a_{5,2} a_{6,7} a_{7,8} a_{8,3} - a_{1,3} a_{2,2} a_{3,7} a_{4,8} a_{5,1} a_{6,5} a_{7,6} a_{8,4} \\
&+ a_{1,2} a_{2,3} a_{3,7} a_{4,8} a_{5,1} a_{6,5} a_{7,6} a_{8,4} - a_{1,1} a_{2,3} a_{3,7} a_{4,8} a_{5,2} a_{6,5} a_{7,6} a_{8,4}
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{1,1} a_{2,2} a_{3,7} a_{4,8} a_{5,3} a_{6,5} a_{7,6} a_{8,4} + a_{1,3} a_{2,2} a_{3,5} a_{4,8} a_{5,1} a_{6,7} a_{7,6} a_{8,4} \\
&- a_{1,2} a_{2,3} a_{3,5} a_{4,8} a_{5,1} a_{6,7} a_{7,6} a_{8,4} + a_{1,1} a_{2,3} a_{3,5} a_{4,8} a_{5,2} a_{6,7} a_{7,6} a_{8,4} \\
&- a_{1,1} a_{2,2} a_{3,5} a_{4,8} a_{5,3} a_{6,7} a_{7,6} a_{8,4} + a_{1,3} a_{2,2} a_{3,6} a_{4,8} a_{5,1} a_{6,5} a_{7,7} a_{8,4} \\
&- a_{1,2} a_{2,3} a_{3,6} a_{4,8} a_{5,1} a_{6,5} a_{7,7} a_{8,4} + a_{1,1} a_{2,3} a_{3,6} a_{4,8} a_{5,2} a_{6,5} a_{7,7} a_{8,4} \\
&- a_{1,1} a_{2,2} a_{3,6} a_{4,8} a_{5,3} a_{6,5} a_{7,7} a_{8,4} - a_{1,3} a_{2,2} a_{3,5} a_{4,8} a_{5,1} a_{6,6} a_{7,7} a_{8,4} \\
&+ a_{1,2} a_{2,3} a_{3,5} a_{4,8} a_{5,1} a_{6,6} a_{7,7} a_{8,4} - a_{1,1} a_{2,3} a_{3,5} a_{4,8} a_{5,2} a_{6,6} a_{7,7} a_{8,4} \\
&+ a_{1,1} a_{2,2} a_{3,5} a_{4,8} a_{5,3} a_{6,6} a_{7,7} a_{8,4} + a_{1,3} a_{2,2} a_{3,7} a_{4,6} a_{5,1} a_{6,5} a_{7,8} a_{8,4} \\
&- a_{1,2} a_{2,3} a_{3,7} a_{4,6} a_{5,1} a_{6,5} a_{7,8} a_{8,4} - a_{1,3} a_{2,2} a_{3,6} a_{4,7} a_{5,1} a_{6,5} a_{7,8} a_{8,4} \\
&+ a_{1,2} a_{2,3} a_{3,6} a_{4,7} a_{5,1} a_{6,5} a_{7,8} a_{8,4} + a_{1,1} a_{2,3} a_{3,7} a_{4,6} a_{5,2} a_{6,5} a_{7,8} a_{8,4} \\
&- a_{1,1} a_{2,3} a_{3,6} a_{4,7} a_{5,2} a_{6,5} a_{7,8} a_{8,4} - a_{1,1} a_{2,2} a_{3,7} a_{4,6} a_{5,3} a_{6,5} a_{7,8} a_{8,4} \\
&+ a_{1,1} a_{2,2} a_{3,6} a_{4,7} a_{5,3} a_{6,5} a_{7,8} a_{8,4} + a_{1,3} a_{2,2} a_{3,5} a_{4,7} a_{5,1} a_{6,6} a_{7,8} a_{8,4} \\
&- a_{1,2} a_{2,3} a_{3,5} a_{4,7} a_{5,1} a_{6,6} a_{7,8} a_{8,4} + a_{1,1} a_{2,3} a_{3,5} a_{4,7} a_{5,2} a_{6,6} a_{7,8} a_{8,4} \\
&- a_{1,1} a_{2,2} a_{3,5} a_{4,7} a_{5,3} a_{6,6} a_{7,8} a_{8,4} - a_{1,3} a_{2,2} a_{3,5} a_{4,6} a_{5,1} a_{6,7} a_{7,8} a_{8,4} \\
&+ a_{1,2} a_{2,3} a_{3,5} a_{4,6} a_{5,1} a_{6,7} a_{7,8} a_{8,4} - a_{1,1} a_{2,3} a_{3,5} a_{4,6} a_{5,2} a_{6,7} a_{7,8} a_{8,4} \\
&+ a_{1,1} a_{2,2} a_{3,5} a_{4,6} a_{5,3} a_{6,7} a_{7,8} a_{8,4} + a_{1,3} a_{2,2} a_{3,7} a_{4,8} a_{5,1} a_{6,4} a_{7,6} a_{8,5} \\
&- a_{1,2} a_{2,3} a_{3,7} a_{4,8} a_{5,1} a_{6,4} a_{7,6} a_{8,5} + a_{1,1} a_{2,3} a_{3,7} a_{4,8} a_{5,2} a_{6,4} a_{7,6} a_{8,5} \\
&- a_{1,1} a_{2,2} a_{3,7} a_{4,8} a_{5,3} a_{6,4} a_{7,6} a_{8,5} - a_{1,3} a_{2,2} a_{3,4} a_{4,8} a_{5,1} a_{6,7} a_{7,6} a_{8,5} \\
&+ a_{1,2} a_{2,3} a_{3,4} a_{4,8} a_{5,1} a_{6,7} a_{7,6} a_{8,5} - a_{1,1} a_{2,3} a_{3,4} a_{4,8} a_{5,2} a_{6,7} a_{7,6} a_{8,5} \\
&+ a_{1,1} a_{2,2} a_{3,4} a_{4,8} a_{5,3} a_{6,7} a_{7,6} a_{8,5} - a_{1,3} a_{2,2} a_{3,6} a_{4,8} a_{5,1} a_{6,4} a_{7,7} a_{8,5} \\
&+ a_{1,2} a_{2,3} a_{3,6} a_{4,8} a_{5,1} a_{6,4} a_{7,7} a_{8,5} - a_{1,1} a_{2,3} a_{3,6} a_{4,8} a_{5,2} a_{6,4} a_{7,7} a_{8,5} \\
&+ a_{1,1} a_{2,2} a_{3,6} a_{4,8} a_{5,3} a_{6,4} a_{7,7} a_{8,5} + a_{1,3} a_{2,2} a_{3,4} a_{4,8} a_{5,1} a_{6,6} a_{7,7} a_{8,5} \\
&- a_{1,2} a_{2,3} a_{3,4} a_{4,8} a_{5,1} a_{6,6} a_{7,7} a_{8,5} + a_{1,1} a_{2,3} a_{3,4} a_{4,8} a_{5,2} a_{6,6} a_{7,7} a_{8,5} \\
&- a_{1,1} a_{2,2} a_{3,4} a_{4,8} a_{5,3} a_{6,6} a_{7,7} a_{8,5} - a_{1,3} a_{2,2} a_{3,7} a_{4,6} a_{5,1} a_{6,4} a_{7,8} a_{8,5} \\
&+ a_{1,2} a_{2,3} a_{3,7} a_{4,6} a_{5,1} a_{6,4} a_{7,8} a_{8,5} + a_{1,3} a_{2,2} a_{3,6} a_{4,7} a_{5,1} a_{6,4} a_{7,8} a_{8,5} \\
&- a_{1,2} a_{2,3} a_{3,6} a_{4,7} a_{5,1} a_{6,4} a_{7,8} a_{8,5} - a_{1,1} a_{2,3} a_{3,7} a_{4,6} a_{5,2} a_{6,4} a_{7,8} a_{8,5} \\
&+ a_{1,1} a_{2,3} a_{3,6} a_{4,7} a_{5,2} a_{6,4} a_{7,8} a_{8,5} + a_{1,1} a_{2,2} a_{3,7} a_{4,6} a_{5,3} a_{6,4} a_{7,8} a_{8,5} \\
&- a_{1,1} a_{2,2} a_{3,6} a_{4,7} a_{5,3} a_{6,4} a_{7,8} a_{8,5} - a_{1,3} a_{2,2} a_{3,4} a_{4,7} a_{5,1} a_{6,6} a_{7,8} a_{8,5} \\
&+ a_{1,2} a_{2,3} a_{3,4} a_{4,7} a_{5,1} a_{6,6} a_{7,8} a_{8,5} - a_{1,1} a_{2,3} a_{3,4} a_{4,7} a_{5,2} a_{6,6} a_{7,8} a_{8,5} \\
&+ a_{1,1} a_{2,2} a_{3,4} a_{4,7} a_{5,3} a_{6,6} a_{7,8} a_{8,5} + a_{1,3} a_{2,2} a_{3,4} a_{4,6} a_{5,1} a_{6,7} a_{7,8} a_{8,5} \\
&- a_{1,2} a_{2,3} a_{3,4} a_{4,6} a_{5,1} a_{6,7} a_{7,8} a_{8,5} + a_{1,1} a_{2,3} a_{3,4} a_{4,6} a_{5,2} a_{6,7} a_{7,8} a_{8,5} \\
&- a_{1,1} a_{2,2} a_{3,4} a_{4,6} a_{5,3} a_{6,7} a_{7,8} a_{8,5} \, .
\end{aligned}
\tag{308}
$$

It is readily apparent that (308) is a rather complicated expression involving 128 terms. In order to simplify the algebra somewhat, we have divided the expression into four distinct subexpressions, such that

$$\text{denom} = \text{denom}_A + \text{denom}_B + \text{denom}_C + \text{denom}_D . \tag{309}$$

where denom_A is the sum of terms involving the element $a_{8,2}$, denom_B is the sum of terms involving the element $a_{8,3}$, denom_C is the sum of terms involving the element $a_{8,4}$, and denom_D is the sum of terms involving the element $a_{8,5}$. This simplification allows us to work with 32 terms at a time, and yields the following *generic expressions* for the various parts:

$$\begin{aligned}\text{denom}_A = a_{8,2} \{a_{1,3}\, a_{5,1} - a_{1,1}\, a_{5,3}\} \big[a_{2,5} \big[(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8})& \\ \times \{a_{3,4}\, a_{6,7} - a_{3,7}\, a_{6,4}\} + (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{a_{3,4}\, a_{6,6} - a_{3,6}\, a_{6,4}\}\big]& \\ - a_{2,4} \big[(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}) \{a_{3,5}\, a_{6,7} - a_{3,7}\, a_{6,5}\}& \\ + (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{a_{3,5}\, a_{6,6} - a_{3,6}\, a_{6,5}\}\big]\big] ,&\end{aligned} \tag{310}$$

$$\begin{aligned}\text{denom}_B = a_{8,3} \{a_{1,1}\, a_{5,2} - a_{1,2}\, a_{5,1}\} \big[a_{2,5} \big[(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8})& \\ \times \{a_{3,4}\, a_{6,7} - a_{3,7}\, a_{6,4}\} + (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{a_{3,4}\, a_{6,6} - a_{3,6}\, a_{6,4}\}\big]& \\ - a_{2,4} \big[(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}) \{a_{3,5}\, a_{6,7} - a_{3,7}\, a_{6,5}\}& \\ + (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{a_{3,5}\, a_{6,6} - a_{3,6}\, a_{6,5}\}\big]\big] ,&\end{aligned} \tag{311}$$

$$\begin{aligned}\text{denom}_C = a_{8,4} \big[(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}) \{a_{3,5}\, a_{6,7} - a_{3,7}\, a_{6,5}\}& \\ + (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{a_{3,5}\, a_{6,6} - a_{3,6}\, a_{6,5}\}\big]& \\ \times \{a_{2,2} (a_{1,3}\, a_{5,1} - a_{1,1}\, a_{5,3}) + a_{2,3} (a_{1,1}\, a_{5,2} - a_{1,2}\, a_{5,1})\} ,&\end{aligned} \tag{312}$$

and

$$\begin{aligned}\text{denom}_D = a_{8,5} \big[(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}) \{a_{3,4}\, a_{6,7} - a_{3,7}\, a_{6,4}\}& \\ + (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{a_{3,4}\, a_{6,6} - a_{3,6}\, a_{6,4}\}\big]& \\ \times \{a_{2,2} (a_{1,1}\, a_{5,3} - a_{1,3}\, a_{5,1}) + a_{2,3} (a_{1,2}\, a_{5,1} - a_{1,1}\, a_{5,2})\} .&\end{aligned} \tag{313}$$

Substituting (310) through (313) into (309) and combining like terms yields the following *generic expression* for the *common denominator*:

$$
\begin{aligned}
\text{denom} = {} & \Big[\Big[\left(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}\right)\left\{a_{3,4}\, a_{6,7} - a_{3,7}\, a_{6,4}\right\} \\
& + \left(a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}\right)\left\{a_{3,4}\, a_{6,6} - a_{3,6}\, a_{6,4}\right\}\Big] \\
& \times \Big[\left(a_{2,5}\, a_{8,2} - a_{2,2}\, a_{8,5}\right)\left(a_{1,3}\, a_{5,1} - a_{1,1}\, a_{5,3}\right) \\
& + \left(a_{2,3}\, a_{8,5} - a_{2,5}\, a_{8,3}\right)\left(a_{1,2}\, a_{5,1} - a_{1,1}\, a_{5,2}\right)\Big]\Big] \\
& - \Big[\Big[\left(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}\right)\left\{a_{3,5}\, a_{6,7} - a_{3,7}\, a_{6,5}\right\} \\
& + \left(a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}\right)\left\{a_{3,5}\, a_{6,6} - a_{3,6}\, a_{6,5}\right\}\Big] \\
& \times \Big[\left(a_{2,4}\, a_{8,2} - a_{2,2}\, a_{8,4}\right)\left(a_{1,3}\, a_{5,1} - a_{1,1}\, a_{5,3}\right) \\
& + \left(a_{2,3}\, a_{8,4} - a_{2,4}\, a_{8,3}\right)\left(a_{1,2}\, a_{5,1} - a_{1,1}\, a_{5,2}\right)\Big]\Big] .
\end{aligned}
\tag{314}
$$

Substituting (277) through (304) into (314), using the facts that $y_S = 0$, $y_{B_1} = D_1$, and $y_{B_2} = D_2$, and performing some algebraic manipulations results in the following *general expression* for the *common denominator*:

$$
\begin{aligned}
\text{denom} = {} & \Big[\Big(\rho_3(D_2)\, Y_3^+(D_2)\, \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2)\, \frac{dY_3^+(D_2)}{dy}\, Y_4^+(D_2)\Big) \\
& \times \Big\{\rho_3(D_1)\, \frac{dY_{2b}^+(D_1)}{dy}\, Y_3^-(D_1) - \rho_2(D_1)\, Y_{2b}^+(D_1)\, \frac{dY_3^-(D_1)}{dy}\Big\} \\
& + \Big(\rho_4(D_2)\, \frac{dY_3^-(D_2)}{dy}\, Y_4^+(D_2) - \rho_3(D_2)\, Y_3^-(D_2)\, \frac{dY_4^+(D_2)}{dy}\Big)
\end{aligned}
$$

$$\times \left\{ \rho_3(D_1) \frac{dY^+_{2b}(D_1)}{dy} Y^+_3(D_1) - \rho_2(D_1)\, Y^+_{2b}(D_1) \frac{dY^+_3(D_1)}{dy} \right\} \Bigg]$$

$$\times \left[\left(Y^+_{2a}(y_o) \frac{dY^-_{2b}(y_o)}{dy} - \frac{dY^+_{2a}(y_o)}{dy} Y^-_{2b}(y_o) \right) \right.$$

$$\times \left\{ \rho_1(0)\, Y^-_1(0) \frac{dY^-_{2a}(0)}{dy} - \rho_2(0) \frac{dY^-_1(0)}{dy} Y^-_{2a}(0) \right\}$$

$$+ \left(\frac{dY^-_{2a}(y_o)}{dy} Y^-_{2b}(y_o) - Y^-_{2a}(y_o) \frac{dY^-_{2b}(y_o)}{dy} \right)$$

$$\left. \times \left\{ \rho_1(0)\, Y^-_1(0) \frac{dY^+_{2a}(0)}{dy} - \rho_2(0) \frac{dY^-_1(0)}{dy} Y^+_{2a}(0) \right\} \right]$$

$$- \left[\left(\rho_3(D_2)\, Y^+_3(D_2) \frac{dY^+_4(D_2)}{dy} - \rho_4(D_2) \frac{dY^+_3(D_2)}{dy} Y^+_4(D_2) \right) \right.$$

$$\times \left\{ \rho_3(D_1) \frac{dY^-_{2b}(D_1)}{dy} Y^-_3(D_1) - \rho_2(D_1)\, Y^-_{2b}(D_1) \frac{dY^-_3(D_1)}{dy} \right\}$$

$$+ \left(\rho_4(D_2) \frac{dY^-_3(D_2)}{dy} Y^+_4(D_2) - \rho_3(D_2)\, Y^-_3(D_2) \frac{dY^+_4(D_2)}{dy} \right)$$

$$\times \left\{ \rho_3(D_1) \frac{dY^-_{2b}(D_1)}{dy} Y^+_3(D_1) - \rho_2(D_1)\, Y^-_{2b}(D_1) \frac{dY^+_3(D_1)}{dy} \right\} \Bigg]$$

$$\times \left[\left(Y^+_{2a}(y_o) \frac{dY^+_{2b}(y_o)}{dy} - \frac{dY^+_{2a}(y_o)}{dy} Y^+_{2b}(y_o) \right) \right.$$

$$\times \left\{ \rho_1(0)\, Y^-_1(0) \frac{dY^-_{2a}(0)}{dy} - \rho_2(0) \frac{dY^-_1(0)}{dy} Y^-_{2a}(0) \right\}$$

$$+ \left(\frac{dY^-_{2a}(y_o)}{dy} Y^+_{2b}(y_o) - Y^-_{2a}(y_o) \frac{dY^+_{2b}(y_o)}{dy} \right)$$

$$\left. \times \left\{ \rho_1(0)\, Y^-_1(0) \frac{dY^+_{2a}(0)}{dy} - \rho_2(0) \frac{dY^-_1(0)}{dy} Y^+_{2a}(0) \right\} \right]. \qquad (315)$$

If we now assume constant speed of sound and constant density in a specific medium, the depth-dependent functions in (315) become complex exponentials and the denominator becomes (*using the subscript* c *to indicate the constant speed of sound assumption*)

$$
\begin{aligned}
\mathrm{denom}_c = 2\,k_{y_2}\, e^{-jk_{y_4} D_2} \Big[& \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \\
& \times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1}\, e^{-jk_{y_3} D_2}\, e^{+jk_{y_3} D_1} \\
& + \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right) \\
& \times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1}\, e^{+jk_{y_3} D_2}\, e^{-jk_{y_3} D_1} \\
& - \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right) \\
& \times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1}\, e^{-jk_{y_3} D_2}\, e^{+jk_{y_3} D_1} \\
& - \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \\
& \times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1}\, e^{+jk_{y_3} D_2}\, e^{-jk_{y_3} D_1} \Big] \, . \qquad (316)
\end{aligned}
$$

Now that the denominator has been simplified, we'll concentrate on obtaining expressions for each of the unknown constants in the order stated above. The first constant is

$$A_{2a} = \frac{num_2}{denom}, \qquad (317)$$

where

$$\begin{aligned}
num_2 = G_1 \Big[&- a_{1,3}\, a_{2,5}\, a_{3,7}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,6} \\
&+ a_{1,1}\, a_{2,5}\, a_{3,7}\, a_{4,8}\, a_{5,3}\, a_{6,4}\, a_{7,6} + a_{1,3}\, a_{2,4}\, a_{3,7}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,6} \\
&- a_{1,1}\, a_{2,4}\, a_{3,7}\, a_{4,8}\, a_{5,3}\, a_{6,5}\, a_{7,6} + a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,7}\, a_{7,6} \\
&- a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,7}\, a_{7,6} - a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,3}\, a_{6,7}\, a_{7,6} \\
&+ a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,3}\, a_{6,7}\, a_{7,6} + a_{1,3}\, a_{2,5}\, a_{3,6}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,7} \\
&- a_{1,1}\, a_{2,5}\, a_{3,6}\, a_{4,8}\, a_{5,3}\, a_{6,4}\, a_{7,7} - a_{1,3}\, a_{2,4}\, a_{3,6}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,7} \\
&+ a_{1,1}\, a_{2,4}\, a_{3,6}\, a_{4,8}\, a_{5,3}\, a_{6,5}\, a_{7,7} - a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,6}\, a_{7,7} \\
&+ a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,6}\, a_{7,7} + a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,3}\, a_{6,6}\, a_{7,7} \\
&- a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,3}\, a_{6,6}\, a_{7,7} + a_{1,3}\, a_{2,5}\, a_{3,7}\, a_{4,6}\, a_{5,1}\, a_{6,4}\, a_{7,8} \\
&- a_{1,3}\, a_{2,5}\, a_{3,6}\, a_{4,7}\, a_{5,1}\, a_{6,4}\, a_{7,8} - a_{1,1}\, a_{2,5}\, a_{3,7}\, a_{4,6}\, a_{5,3}\, a_{6,4}\, a_{7,8} \\
&+ a_{1,1}\, a_{2,5}\, a_{3,6}\, a_{4,7}\, a_{5,3}\, a_{6,4}\, a_{7,8} - a_{1,3}\, a_{2,4}\, a_{3,7}\, a_{4,6}\, a_{5,1}\, a_{6,5}\, a_{7,8} \\
&+ a_{1,3}\, a_{2,4}\, a_{3,6}\, a_{4,7}\, a_{5,1}\, a_{6,5}\, a_{7,8} + a_{1,1}\, a_{2,4}\, a_{3,7}\, a_{4,6}\, a_{5,3}\, a_{6,5}\, a_{7,8} \\
&- a_{1,1}\, a_{2,4}\, a_{3,6}\, a_{4,7}\, a_{5,3}\, a_{6,5}\, a_{7,8} + a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,7}\, a_{5,1}\, a_{6,6}\, a_{7,8} \\
&- a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,7}\, a_{5,1}\, a_{6,6}\, a_{7,8} - a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,7}\, a_{5,3}\, a_{6,6}\, a_{7,8} \\
&+ a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,7}\, a_{5,3}\, a_{6,6}\, a_{7,8} - a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,6}\, a_{5,1}\, a_{6,7}\, a_{7,8} \\
&+ a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,6}\, a_{5,1}\, a_{6,7}\, a_{7,8} + a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,6}\, a_{5,3}\, a_{6,7}\, a_{7,8} \\
&- a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,6}\, a_{5,3}\, a_{6,7}\, a_{7,8} \Big]. \qquad (318)
\end{aligned}$$

Factoring (318) and collecting common terms yields the following *generic expression* for the *numerator* of A_{2a}:

$$\begin{aligned}
num_2 = G_1 \left(a_{1,3}\, a_{5,1} - a_{1,1}\, a_{5,3}\right) \Big[a_{2,5} \Big[&\left(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}\right) \\
\times \left\{a_{3,4}\, a_{6,7} - a_{3,7}\, a_{6,4}\right\} + &\left(a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}\right) \left\{a_{3,4}\, a_{6,6} - a_{3,6}\, a_{6,4}\right\} \Big] \Big]
\end{aligned}$$

$$
\begin{aligned}
&- a_{2,4} \Big[\left(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8} \right) \left\{ a_{3,5}\, a_{6,7} - a_{3,7}\, a_{6,5} \right\} \\
&+ \left(a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7} \right) \left\{ a_{3,5}\, a_{6,6} - a_{3,6}\, a_{6,5} \right\} \Big] \Big] . \qquad (319)
\end{aligned}
$$

Substituting appropriate expressions into (319) and using (307) yields the following *general expression* for the *numerator* of A_{2a}:

$$
num_2 = \frac{-k_r}{2\pi} \left(\rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^-(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^-(0) \right)
$$

$$
\times \Big[- Y_{2b}^-(y_o) \Big[\left(\rho_3(D_2)\, Y_3^+(D_2) \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2) \frac{dY_3^+(D_2)}{dy} Y_4^+(D_2) \right)
$$

$$
\times \left\{ \rho_3(D_1) \frac{dY_{2b}^+(D_1)}{dy} Y_3^-(D_1) - \rho_2(D_1)\, Y_{2b}^+(D_1) \frac{dY_3^-(D_1)}{dy} \right\}
$$

$$
+ \left(\rho_4(D_2) \frac{dY_3^-(D_2)}{dy} Y_4^+(D_2) - \rho_3(D_2)\, Y_3^-(D_2) \frac{dY_4^+(D_2)}{dy} \right)
$$

$$
\times \left\{ \rho_3(D_1) \frac{dY_{2b}^+(D_1)}{dy} Y_3^+(D_1) - \rho_2(D_1)\, Y_{2b}^+(D_1) \frac{dY_3^+(D_1)}{dy} \right\} \Big]
$$

$$
+ Y_{2b}^+(y_o) \Big[\left(\rho_3(D_2)\, Y_3^+(D_2) \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2) \frac{dY_3^+(D_2)}{dy} Y_4^+(D_2) \right)
$$

$$\times \left\{ \rho_3(D_1) \frac{dY_{2b}^-(D_1)}{dy} Y_3^-(D_1) - \rho_2(D_1)\, Y_{2b}^-(D_1) \frac{dY_3^-(D_1)}{dy} \right\}$$

$$+ \left(\rho_4(D_2) \frac{dY_3^-(D_2)}{dy} Y_4^+(D_2) - \rho_3(D_2)\, Y_3^-(D_2) \frac{dY_4^+(D_2)}{dy} \right)$$

$$\times \left\{ \rho_3(D_1) \frac{dY_{2b}^-(D_1)}{dy} Y_3^+(D_1) - \rho_2(D_1)\, Y_{2b}^-(D_1) \frac{dY_3^+(D_1)}{dy} \right\} \Big] \Big] . \qquad (320)$$

Once again, making the constant speed of sound and constant density assumptions and substituting the appropriate depth-dependent expressions allows us to write the numerator of A_{2a} as (*using the subscript* c *to indicate the constant speed of sound assumption*)

$$num_{2_c} = \frac{-jk_r}{2\pi} \left(\rho_1\, k_{y_2} - \rho_2\, k_{y_1} \right) e^{-jk_{y_4} D_2}$$

$$\times \Big[\left(\rho_3\, k_{y_2} + \rho_2\, k_{y_3} \right) \left(\rho_3\, k_{y_4} - \rho_4\, k_{y_3} \right) e^{-jk_{y_2} D_1}\, e^{+jk_{y_2} y_0}\, e^{-jk_{y_3} D_2}\, e^{+jk_{y_3} D_1}$$

$$- \left(\rho_3\, k_{y_2} - \rho_2\, k_{y_3} \right) \left(\rho_3\, k_{y_4} + \rho_4\, k_{y_3} \right) e^{-jk_{y_2} D_1}\, e^{+jk_{y_2} y_0}\, e^{+jk_{y_3} D_2}\, e^{-jk_{y_3} D_1}$$

$$+\left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)\left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{+jk_{y_2}D_1} e^{-jk_{y_2}y_0} e^{-jk_{y_3}D_2} e^{+jk_{y_3}D_1}$$

$$-\left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)\left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{+jk_{y_2}D_1} e^{-jk_{y_2}y_0} e^{+jk_{y_3}D_2} e^{-jk_{y_3}D_1}\Big] .$$

(321)

Thus, the *general result* for A_{2a} is formed by dividing (320) by (315) and is given by

$$A_{2a} = \frac{-k_r}{2\pi}\left(\rho_1(0)\, Y_1^-(0)\frac{dY_{2a}^-(0)}{dy} - \rho_2(0)\frac{dY_1^-(0)}{dy} Y_{2a}^-(0)\right)$$

$$\times\Big[- Y_{2b}^-(y_o)\Big[\left(\rho_3(D_2)\, Y_3^+(D_2)\frac{dY_4^+(D_2)}{dy} - \rho_4(D_2)\frac{dY_3^+(D_2)}{dy} Y_4^+(D_2)\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^+(D_1)}{dy} Y_3^-(D_1) - \rho_2(D_1)\, Y_{2b}^+(D_1)\frac{dY_3^-(D_1)}{dy}\right\}$$

$$+\left(\rho_4(D_2)\frac{dY_3^-(D_2)}{dy} Y_4^+(D_2) - \rho_3(D_2)\, Y_3^-(D_2)\frac{dY_4^+(D_2)}{dy}\right)$$

$$\times \left\{\rho_3(D_1)\frac{dY^{+}_{2b}(D_1)}{dy}Y^{+}_3(D_1) - \rho_2(D_1)\,Y^{+}_{2b}(D_1)\frac{dY^{+}_3(D_1)}{dy}\right\}\Big]$$

$$+\,Y^{+}_{2b}(y_o)\Big[\Big(\rho_3(D_2)\,Y^{+}_3(D_2)\frac{dY^{+}_4(D_2)}{dy} - \rho_4(D_2)\frac{dY^{+}_3(D_2)}{dy}Y^{+}_4(D_2)\Big)$$

$$\times \left\{\rho_3(D_1)\frac{dY^{-}_{2b}(D_1)}{dy}Y^{-}_3(D_1) - \rho_2(D_1)\,Y^{-}_{2b}(D_1)\frac{dY^{-}_3(D_1)}{dy}\right\}$$

$$+\Big(\rho_4(D_2)\frac{dY^{-}_3(D_2)}{dy}Y^{+}_4(D_2) - \rho_3(D_2)\,Y^{-}_3(D_2)\frac{dY^{+}_4(D_2)}{dy}\Big)$$

$$\times \left\{\rho_3(D_1)\frac{dY^{-}_{2b}(D_1)}{dy}Y^{+}_3(D_1) - \rho_2(D_1)\,Y^{-}_{2b}(D_1)\frac{dY^{+}_3(D_1)}{dy}\right\}\Big]\Big]\;/$$

$$\Big[\Big[\Big(\rho_3(D_2)\,Y^{+}_3(D_2)\frac{dY^{+}_4(D_2)}{dy} - \rho_4(D_2)\frac{dY^{+}_3(D_2)}{dy}Y^{+}_4(D_2)\Big)$$

$$\times \left\{\rho_3(D_1)\frac{dY^{+}_{2b}(D_1)}{dy}Y^{-}_3(D_1) - \rho_2(D_1)\,Y^{+}_{2b}(D_1)\frac{dY^{-}_3(D_1)}{dy}\right\}$$

$$+\Big(\rho_4(D_2)\frac{dY^{-}_3(D_2)}{dy}Y^{+}_4(D_2) - \rho_3(D_2)\,Y^{-}_3(D_2)\frac{dY^{+}_4(D_2)}{dy}\Big)$$

$$\times \left\{\rho_3(D_1) \frac{dY^+_{2b}(D_1)}{dy} Y^+_3(D_1) - \rho_2(D_1)\, Y^+_{2b}(D_1) \frac{dY^+_3(D_1)}{dy}\right\}\Bigg]$$

$$\times \left[\left(Y^+_{2a}(y_o) \frac{dY^-_{2b}(y_o)}{dy} - \frac{dY^+_{2a}(y_o)}{dy} Y^-_{2b}(y_o)\right)\right.$$

$$\times \left\{\rho_1(0)\, Y^-_1(0) \frac{dY^-_{2a}(0)}{dy} - \rho_2(0) \frac{dY^-_1(0)}{dy} Y^-_{2a}(0)\right\}$$

$$+ \left(\frac{dY^-_{2a}(y_o)}{dy} Y^-_{2b}(y_o) - Y^-_{2a}(y_o) \frac{dY^-_{2b}(y_o)}{dy}\right)$$

$$\left.\times \left\{\rho_1(0)\, Y^-_1(0) \frac{dY^+_{2a}(0)}{dy} - \rho_2(0) \frac{dY^-_1(0)}{dy} Y^+_{2a}(0)\right\}\right]$$

$$- \left[\left(\rho_3(D_2)\, Y^+_3(D_2) \frac{dY^+_4(D_2)}{dy} - \rho_4(D_2) \frac{dY^+_3(D_2)}{dy} Y^+_4(D_2)\right)\right.$$

$$\times \left\{\rho_3(D_1) \frac{dY^-_{2b}(D_1)}{dy} Y^-_3(D_1) - \rho_2(D_1)\, Y^-_{2b}(D_1) \frac{dY^-_3(D_1)}{dy}\right\}$$

$$+ \left(\rho_4(D_2) \frac{dY^-_3(D_2)}{dy} Y^+_4(D_2) - \rho_3(D_2)\, Y^-_3(D_2) \frac{dY^+_4(D_2)}{dy}\right)$$

$$\times \left\{\rho_3(D_1)\frac{dY^-_{2b}(D_1)}{dy}Y^+_3(D_1) - \rho_2(D_1)\,Y^-_{2b}(D_1)\frac{dY^+_3(D_1)}{dy}\right\}\Big]$$

$$\times \Big[\Big(Y^+_{2a}(y_o)\frac{dY^+_{2b}(y_o)}{dy} - \frac{dY^+_{2a}(y_o)}{dy}Y^+_{2b}(y_o)\Big)$$

$$\times \left\{\rho_1(0)\,Y^-_1(0)\frac{dY^-_{2a}(0)}{dy} - \rho_2(0)\frac{dY^-_1(0)}{dy}Y^-_{2a}(0)\right\}$$

$$+ \Big(\frac{dY^-_{2a}(y_o)}{dy}Y^+_{2b}(y_o) - Y^-_{2a}(y_o)\frac{dY^+_{2b}(y_o)}{dy}\Big)$$

$$\times \left\{\rho_1(0)\,Y^-_1(0)\frac{dY^+_{2a}(0)}{dy} - \rho_2(0)\frac{dY^-_1(0)}{dy}Y^+_{2a}(0)\right\}\Big]\Big]\,. \qquad (322)$$

For constant speed of sound and ambient density, A_{2a} reduces to

$$A_{2a_c} = \frac{-jk_r}{2\pi}\left(\rho_1\,k_{y_2} - \rho_2\,k_{y_1}\right)e^{-jk_{y_4}D_2}$$

$$\times \Big[\Big(\rho_3\,k_{y_2} + \rho_2\,k_{y_3}\Big)\Big(\rho_3\,k_{y_4} - \rho_4\,k_{y_3}\Big)\,e^{-jk_{y_2}D_1}\,e^{+jk_{y_2}y_o}\,e^{-jk_{y_3}D_2}\,e^{+jk_{y_3}D_1}$$

$$- \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_2} y_o} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$+ \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_2} y_o} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_2} y_o} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big] /$$

$$\Big[2 k_{y_2} e^{-jk_{y_4} D_2} \Big[\left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1}\, e^{-jk_{y_3} D_2}\, e^{+jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1}\, e^{+jk_{y_3} D_2}\, e^{-jk_{y_3} D_1}\Big]\Big] . \qquad (323)$$

Eliminating the common term $e^{-jk_{y_4} D_2}$, dividing numerator and denominator by $\left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right)$, and using the definition of R_{21} given by (238) yields

$$A_{2a_c} = \frac{-jk_r}{4\pi k_{y_2}} R_{21}$$

$$\times \Big[\left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)\left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{-jk_{y_2} D_1}\, e^{+jk_{y_2} y_0}\, e^{-jk_{y_3} D_2}\, e^{+jk_{y_3} D_1}$$

$$- \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_2} y_0} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$+ \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_2} y_0} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_2} y_0} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big] \Big/$$

$$\Big[\left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right) \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$- R_{21} \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right) \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- R_{21} \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big] . \qquad (324)$$

Dividing numerator and denominator of (324) by $\left(\rho_3 k_{y_2} + \rho_2 k_{y_4}\right)$, and using the definition of R_{23} given by (241) yields

$$A_{2a_c} = \frac{-jk_r}{4\pi k_{y_2}} R_{21} \Big[\left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_2} y_0} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- R_{23} \left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_2} y_0} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$+ R_{23} \left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_2} y_0} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_2} y_0} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big] \Big/$$

$$\Big[R_{23} \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$- R_{21} \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- R_{21} R_{23} \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big] . \qquad (325)$$

Dividing numerator and denominator of (325) by $\left(\rho_4 \; k_{y_3} + \rho_3 \; k_{y_4}\right)$, defining a reflection coefficient at the boundary between medium three and medium four, R_{34}, as

$$R_{34} = \frac{\left(\rho_4 \; k_{y_3} - \rho_3 \; k_{y_4}\right)}{\left(\rho_4 \; k_{y_3} + \rho_3 \; k_{y_4}\right)}, \qquad (326)$$

and multiplying through by the -1 appearing in the numerator yields the following desired expression:

$$A_{2a_c} = \frac{+jk_r}{4\pi k_{y_2}} R_{21} \Big[R_{34} \, e^{-jk_{y_2}(D_1 - y_0)} e^{-jk_{y_3}(D_2 - D_1)}$$

$$+ R_{23}\, e^{-jk_{y_2}(D_1 - y_o)}\, e^{-jk_{y_3}(D_2 - D_1)}$$

$$+ R_{23} R_{34}\, e^{+jk_{y_2}(D_1 - y_o)}\, e^{-jk_{y_3}(D_2 - D_1)} + e^{+jk_{y_2}(D_1 - y_o)}\, e^{+jk_{y_3}(D_2 - D_1)} \Big] \Big/$$

$$\Big[R_{23} R_{34}\, e^{-jk_{y_3}(D_2 - D_1)}\, e^{+jk_{y_2} D_1} + e^{+jk_{y_3}(D_2 - D_1)}\, e^{+jk_{y_2} D_1}$$

$$- R_{21} R_{34}\, e^{-jk_{y_3}(D_2 - D_1)}\, e^{-jk_{y_2} D_1} - R_{21} R_{23}\, e^{+jk_{y_3}(D_2 - D_1)}\, e^{-jk_{y_2} D_1} \Big] . \quad (327)$$

Now that we have derived the equivalent classical expression for the four media waveguide problem for the unknown constant A_{2a_c}, we must verify that it reduces to the proper expression if the fourth medium is removed mathematically. In order to conduct this evaluation, we must assume the following:

$$D_2 = D_1 = D \text{ (that is, } y_{B_2} = y_{B_1} = y_B \text{)} , \quad (328)$$

and

$$R_{34} = 0 . \quad (329)$$

Substituting conditions (328) and (329) into (327) reveals

$$A_{2a_c} = \frac{+jk_f}{4\pi k_{y_2}} R_{21} \left[R_{23}\, e^{-jk_{y_2}(D - y_0)} + e^{+jk_{y_2}(D - y_0)} \right] \Big/$$

$$\left[e^{+jk_{y_2} D} - R_{21} R_{23}\, e^{-jk_{y_2} D} \right]. \tag{330}$$

Equation (330) is equal to equation (242). Since (242) has already been verified, we can conclude that our derivation of the solutions for $A_{2a'}$ and hence, A_{2a_c} for the four media waveguide problem is correct.

Continuing with the analysis of the four media waveguide problem code results, for the unknown constant B_{2a}, *Mathematica* provided the following numerator:

$$\begin{aligned}
num_3 = G_1 \big[& a_{1,2}\, a_{2,5}\, a_{3,7}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,6} \\
& - a_{1,1}\, a_{2,5}\, a_{3,7}\, a_{4,8}\, a_{5,2}\, a_{6,4}\, a_{7,6} - a_{1,2}\, a_{2,4}\, a_{3,7}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,6} \\
& + a_{1,1}\, a_{2,4}\, a_{3,7}\, a_{4,8}\, a_{5,2}\, a_{6,5}\, a_{7,6} - a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,7}\, a_{7,6} \\
& + a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,7}\, a_{7,6} + a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,2}\, a_{6,7}\, a_{7,6} \\
& - a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,2}\, a_{6,7}\, a_{7,6} - a_{1,2}\, a_{2,5}\, a_{3,6}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,7} \\
& + a_{1,1}\, a_{2,5}\, a_{3,6}\, a_{4,8}\, a_{5,2}\, a_{6,4}\, a_{7,7} + a_{1,2}\, a_{2,4}\, a_{3,6}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,7} \\
& - a_{1,1}\, a_{2,4}\, a_{3,6}\, a_{4,8}\, a_{5,2}\, a_{6,5}\, a_{7,7} + a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,6}\, a_{7,7} \\
& - a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,6}\, a_{7,7} - a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,2}\, a_{6,6}\, a_{7,7} \\
& + a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,2}\, a_{6,6}\, a_{7,7} - a_{1,2}\, a_{2,5}\, a_{3,7}\, a_{4,6}\, a_{5,1}\, a_{6,4}\, a_{7,8} \\
& + a_{1,2}\, a_{2,5}\, a_{3,6}\, a_{4,7}\, a_{5,1}\, a_{6,4}\, a_{7,8} + a_{1,1}\, a_{2,5}\, a_{3,7}\, a_{4,6}\, a_{5,2}\, a_{6,4}\, a_{7,8} \\
& - a_{1,1}\, a_{2,5}\, a_{3,6}\, a_{4,7}\, a_{5,2}\, a_{6,4}\, a_{7,8} + a_{1,2}\, a_{2,4}\, a_{3,7}\, a_{4,6}\, a_{5,1}\, a_{6,5}\, a_{7,8} \\
& - a_{1,2}\, a_{2,4}\, a_{3,6}\, a_{4,7}\, a_{5,1}\, a_{6,5}\, a_{7,8} - a_{1,1}\, a_{2,4}\, a_{3,7}\, a_{4,6}\, a_{5,2}\, a_{6,5}\, a_{7,8} \\
& + a_{1,1}\, a_{2,4}\, a_{3,6}\, a_{4,7}\, a_{5,2}\, a_{6,5}\, a_{7,8} - a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,7}\, a_{5,1}\, a_{6,6}\, a_{7,8} \\
& + a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,7}\, a_{5,1}\, a_{6,6}\, a_{7,8} + a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,7}\, a_{5,2}\, a_{6,6}\, a_{7,8}
\end{aligned}$$

$$- a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,7}\, a_{5,2}\, a_{6,6}\, a_{7,8} + a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,6}\, a_{5,1}\, a_{6,7}\, a_{7,8}$$
$$- a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,6}\, a_{5,1}\, a_{6,7}\, a_{7,8} - a_{1,1}\, a_{2,5}\, a_{3,4}\, a_{4,6}\, a_{5,2}\, a_{6,7}\, a_{7,8}$$
$$+ a_{1,1}\, a_{2,4}\, a_{3,5}\, a_{4,6}\, a_{5,2}\, a_{6,7}\, a_{7,8}\Big] . \qquad (331)$$

Factoring (331) and collecting common terms yields the following *generic expression* for the *numerator* of B_{2a}:

$$\begin{aligned} num_3 = G_1\, (a_{1,1}\, a_{5,2} - a_{1,2}\, a_{5,1}) \Big[a_{2,5} \Big[(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}) \\ \times \{a_{3,4}\, a_{6,7} - a_{3,7}\, a_{6,4}\} + (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{a_{3,4}\, a_{6,6} - a_{3,6}\, a_{6,4}\}\Big] \\ - a_{2,4} \Big[(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}) \{a_{3,5}\, a_{6,7} - a_{3,7}\, a_{6,5}\} \\ + (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{a_{3,5}\, a_{6,6} - a_{3,6}\, a_{6,5}\}\Big]\Big] . \qquad (332) \end{aligned}$$

Substituting appropriate expressions into (332) and using (315) yields the following *general expression* for B_{2a}:

$$B_{2a} = \frac{-k_r}{2\pi} \left(\rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(0) - \rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^+(0)}{dy}\right)$$

$$\times \left[- Y_{2b}^-(y_o) \left[\left(\rho_3(D_2)\, Y_3^+(D_2) \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2) \frac{dY_3^+(D_2)}{dy} Y_4^+(D_2)\right)\right.\right.$$

$$\times \left\{\rho_3(D_1) \frac{dY_{2b}^+(D_1)}{dy} Y_3^-(D_1) - \rho_2(D_1)\, Y_{2b}^+(D_1) \frac{dY_3^-(D_1)}{dy}\right\}$$

$$+ \left(\rho_4(D_2) \frac{dY_3^-(D_2)}{dy} Y_4^+(D_2) - \rho_3(D_2)\, Y_3^-(D_2) \frac{dY_4^+(D_2)}{dy}\right)$$

$$\times \left\{\rho_3(D_1)\frac{dY^+_{2b}(D_1)}{dy}Y^+_3(D_1) - \rho_2(D_1)\,Y^+_{2b}(D_1)\frac{dY^+_3(D_1)}{dy}\right\}\Big]$$

$$+ Y^+_{2b}(y_o)\Big[\Big(\rho_3(D_2)\,Y^+_3(D_2)\frac{dY^+_4(D_2)}{dy} - \rho_4(D_2)\frac{dY^+_3(D_2)}{dy}Y^+_4(D_2)\Big)$$

$$\times \left\{\rho_3(D_1)\frac{dY^-_{2b}(D_1)}{dy}Y^-_3(D_1) - \rho_2(D_1)\,Y^-_{2b}(D_1)\frac{dY^-_3(D_1)}{dy}\right\}$$

$$+ \Big(\rho_4(D_2)\frac{dY^-_3(D_2)}{dy}Y^+_4(D_2) - \rho_3(D_2)\,Y^-_3(D_2)\frac{dY^+_4(D_2)}{dy}\Big)$$

$$\times \left\{\rho_3(D_1)\frac{dY^-_{2b}(D_1)}{dy}Y^+_3(D_1) - \rho_2(D_1)\,Y^-_{2b}(D_1) - \frac{dY^+_3(D_1)}{dy}\right\}\Big]\Big] \Big/$$

$$\Big[\Big[\Big(\rho_3(D_2)\,Y^+_3(D_2)\frac{dY^+_4(D_2)}{dy} - \rho_4(D_2)\frac{dY^+_3(D_2)}{dy}Y^+_4(D_2)\Big)$$

$$\times \left\{\rho_3(D_1)\frac{dY^+_{2b}(D_1)}{dy}Y^-_3(D_1) - \rho_2(D_1)\,Y^+_{2b}(D_1)\frac{dY^-_3(D_1)}{dy}\right\}$$

$$+ \Big(\rho_4(D_2)\frac{dY^-_3(D_2)}{dy}Y^+_4(D_2) - \rho_3(D_2)\,Y^-_3(D_2)\frac{dY^+_4(D_2)}{dy}\Big)$$

$$\times \left\{\rho_3(D_1)\frac{dY^+_{2b}(D_1)}{dy}Y^+_3(D_1) - \rho_2(D_1)\,Y^+_{2b}(D_1)\frac{dY^+_3(D_1)}{dy}\right\}\Big]$$

$$\times \Big[\Big(Y^+_{2a}(y_0)\frac{dY^-_{2b}(y_0)}{dy} - \frac{dY^+_{2a}(y_0)}{dy}Y^-_{2b}(y_0)\Big)$$

$$\times \left\{\rho_1(0)\,Y^-_1(0)\frac{dY^-_{2a}(0)}{dy} - \rho_2(0)\frac{dY^-_1(0)}{dy}Y^-_{2a}(0)\right\}$$

$$+ \Big(\frac{dY^-_{2a}(y_0)}{dy}Y^-_{2b}(y_0) - Y^-_{2a}(y_0)\frac{dY^-_{2b}(y_0)}{dy}\Big)$$

$$\times \left\{\rho_1(0)\,Y^-_1(0)\frac{dY^+_{2a}(0)}{dy} - \rho_2(0)\frac{dY^-_1(0)}{dy}Y^+_{2a}(0)\right\}\Big]$$

$$- \Big[\Big(\rho_3(D_2)\,Y^+_3(D_2)\frac{dY^+_4(D_2)}{dy} - \rho_4(D_2)\frac{dY^+_3(D_2)}{dy}Y^+_4(D_2)\Big)$$

$$\times \left\{\rho_3(D_1)\frac{dY^-_{2b}(D_1)}{dy}Y^-_3(D_1) - \rho_2(D_1)\,Y^-_{2b}(D_1)\frac{dY^-_3(D_1)}{dy}\right\}$$

$$+ \Big(\rho_4(D_2)\frac{dY^-_3(D_2)}{dy}Y^+_4(D_2) - \rho_3(D_2)\,Y^-_3(D_2)\frac{dY^+_4(D_2)}{dy}\Big)$$

$$\times \left\{\rho_3(D_1) \frac{dY_{2b}^-(D_1)}{dy} Y_3^+(D_1) - \rho_2(D_1) Y_{2b}^-(D_1) \frac{dY_3^+(D_1)}{dy}\right\}\Big]$$

$$\times \Big[\left(Y_{2a}^+(y_0) \frac{dY_{2b}^+(y_0)}{dy} - \frac{dY_{2a}^+(y_0)}{dy} Y_{2b}^+(y_0)\right)$$

$$\times \left\{\rho_1(0) Y_1^-(0) \frac{dY_{2a}^-(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^-(0)\right\}$$

$$+ \left(\frac{dY_{2a}^-(y_0)}{dy} Y_{2b}^+(y_0) - Y_{2a}^-(y_0) \frac{dY_{2b}^+(y_0)}{dy}\right)$$

$$\times \left\{\rho_1(0) Y_1^-(0) \frac{dY_{2a}^+(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(0)\right\}\Big]\Big] . \qquad (333)$$

For constant speed of sound and ambient density, (333) becomes

$$B_{2a_c} = \frac{-jk_r}{2\pi} \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) e^{-jk_{y_4} D_2}$$

$$\times \Big[\left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_2} y_0} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)\left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_2} y_0} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$+ \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)\left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_2} y_0} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)\left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_2} y_0} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big] \Big/$$

$$\Big[2 k_{y_2} e^{-jk_{y_4} D_2} \Big[\left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)$$

$$\times\left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$-\left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times\left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}\Big] . \qquad (334)$$

Using the definitions of R_{21}, R_{23}, and R_{34} presented earlier, (334) may be reduced to

$$B_{2a_c} = \frac{+jk_r}{4\pi k_{y_2}}\Big[R_{34}\, e^{-jk_{y_2}(D_1 - y_o)} e^{-jk_{y_3}(D_2 - D_1)}$$

$$+ R_{23}\, e^{-jk_{y_2}(D_1 - y_o)} e^{+jk_{y_3}(D_2 - D_1)}$$

$$+ R_{23} R_{34}\, e^{+jk_{y_2}(D_1 - y_o)} e^{-jk_{y_3}(D_2 - D_1)} + e^{+jk_{y_2}(D_1 - y_o)} e^{+jk_{y_3}(D_2 - D_1)}\Big] /$$

$$\left[R_{23} R_{34} e^{+jk_{y_2} D_1} e^{-jk_{y_3}(D_2 - D_1)} + e^{+jk_{y_2} D_1} e^{+jk_{y_3}(D_2 - D_1)}\right.$$

$$\left. - R_{21} R_{34} e^{-jk_{y_2} D_1} e^{-jk_{y_3}(D_2 - D_1)} - R_{21} e^{-jk_{y_2} D_1} R_{23} e^{+jk_{y_3}(D_2 - D_1)}\right]. \quad (335)$$

Using conditions (328) and (329), (335) may be reduced to

$$B_{2a_c} = \frac{+jk_r}{4\pi k_{y_2}} \left[e^{+jk_{y_2}(D_1 - y_o)} + R_{23} e^{-jk_{y_2}(D_1 - y_o)}\right] /$$

$$\left[e^{+jk_{y_2} D} - R_{21} R_{23} e^{-jk_{y_2} D}\right]. \quad (336)$$

Factoring $e^{-jk_{y_2} y_o}$ out of the numerator of (336), and dividing the numerator and denominator of (336) by $e^{+jk_{y_2} D}$ reveals

$$B_{2a_c} = \frac{+jk_r}{4\pi k_{y_2}} \left[\left(1 + R_{23}\, e^{-j2k_{y_2}(D - y_o)}\right)\right] e^{-jk_{y_2} y_o} \Big/$$

$$\left[1 - R_{21}\, R_{23}\, e^{-j2k_{y_2} D}\right]. \tag{337}$$

Equation (337) is equal to equation (250). Since (250) has already been verified, we can conclude that our derivation of the solutions for B_{2a}, and hence, for B_{2a_c} for the four media waveguide problem is correct.

For the unknown constant A_{2b}, *Mathematica* provided the following numerator:

$$\begin{aligned}
num_4 = G_1 \big(& - a_{1,3}\, a_{2,2}\, a_{3,7}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,6} + \\
& + a_{1,2}\, a_{2,3}\, a_{3,7}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,6} - a_{1,1}\, a_{2,3}\, a_{3,7}\, a_{4,8}\, a_{5,2}\, a_{6,5}\, a_{7,6} \\
& + a_{1,1}\, a_{2,2}\, a_{3,7}\, a_{4,8}\, a_{5,3}\, a_{6,5}\, a_{7,6} + a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,7}\, a_{7,6} \\
& - a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,7}\, a_{7,6} + a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,8}\, a_{5,2}\, a_{6,7}\, a_{7,6} \\
& - a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,8}\, a_{5,3}\, a_{6,7}\, a_{7,6} + a_{1,3}\, a_{2,2}\, a_{3,6}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,7} \\
& - a_{1,2}\, a_{2,3}\, a_{3,6}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,7} + a_{1,1}\, a_{2,3}\, a_{3,6}\, a_{4,8}\, a_{5,2}\, a_{6,5}\, a_{7,7} \\
& - a_{1,1}\, a_{2,2}\, a_{3,6}\, a_{4,8}\, a_{5,3}\, a_{6,5}\, a_{7,7} - a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,6}\, a_{7,7} \\
& + a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,6}\, a_{7,7} - a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,8}\, a_{5,2}\, a_{6,6}\, a_{7,7} \\
& + a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,8}\, a_{5,3}\, a_{6,6}\, a_{7,7} + a_{1,3}\, a_{2,2}\, a_{3,7}\, a_{4,6}\, a_{5,1}\, a_{6,5}\, a_{7,8} \\
& - a_{1,2}\, a_{2,3}\, a_{3,7}\, a_{4,6}\, a_{5,1}\, a_{6,5}\, a_{7,8} - a_{1,3}\, a_{2,2}\, a_{3,6}\, a_{4,7}\, a_{5,1}\, a_{6,5}\, a_{7,8} \\
& + a_{1,2}\, a_{2,3}\, a_{3,6}\, a_{4,7}\, a_{5,1}\, a_{6,5}\, a_{7,8} + a_{1,1}\, a_{2,3}\, a_{3,7}\, a_{4,6}\, a_{5,2}\, a_{6,5}\, a_{7,8} \\
& - a_{1,1}\, a_{2,3}\, a_{3,6}\, a_{4,7}\, a_{5,2}\, a_{6,5}\, a_{7,8} - a_{1,1}\, a_{2,2}\, a_{3,7}\, a_{4,6}\, a_{5,3}\, a_{6,5}\, a_{7,8} \\
& + a_{1,1}\, a_{2,2}\, a_{3,6}\, a_{4,7}\, a_{5,3}\, a_{6,5}\, a_{7,8} + a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,7}\, a_{5,1}\, a_{6,6}\, a_{7,8} \\
& - a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,7}\, a_{5,1}\, a_{6,6}\, a_{7,8} + a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,7}\, a_{5,2}\, a_{6,6}\, a_{7,8} \\
& - a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,7}\, a_{5,3}\, a_{6,6}\, a_{7,8} - a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,6}\, a_{5,1}\, a_{6,7}\, a_{7,8}
\end{aligned}$$

$$+ a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,6}\, a_{5,1}\, a_{6,7}\, a_{7,8} - a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,6}\, a_{5,2}\, a_{6,7}\, a_{7,8} + a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,6}\, a_{5,3}\, a_{6,7}\, a_{7,8} \Big] . \qquad (338)$$

Factoring (338) and collecting common terms yields the following *generic expression* for the *numerator* of A_{2b}:

$$\begin{aligned} num_4 = G_1 \Big[&(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}) \{ a_{3,5}\, a_{6,7} - a_{3,7}\, a_{6,5} \} \\ &+ (a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}) \{ a_{3,5}\, a_{6,6} - a_{3,6}\, a_{6,5} \} \Big] \\ &\times \big[a_{2,2} (a_{1,3}\, a_{5,1} - a_{1,1}\, a_{5,3}) + a_{2,3} (a_{1,1}\, a_{5,2} - a_{1,2}\, a_{5,1}) \big] \end{aligned} \qquad (339)$$

Substituting appropriate expressions into (339) and using (315) yields the following *general expression* for A_{2b}:

$$\begin{aligned} A_{2b} = \frac{-k_r}{2\pi} \Big[\Big[&\Big(\rho_3(D_2)\, Y_3^+(D_2) \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2) \frac{dY_3^+(D_2)}{dy} Y_4^+(D_2) \Big) \\ &\times \Big\{ \rho_3(D_1) \frac{dY_{2b}^-(D_1)}{dy} Y_3^-(D_1) - \rho_2(D_1)\, Y_{2b}^-(D_1) \frac{dY_3^-(D_1)}{dy} \Big\} \\ &+ \Big(\rho_4(D_2) \frac{dY_3^-(D_2)}{dy} Y_4^+(D_2) - \rho_3(D_2)\, Y_3^-(D_2) \frac{dY_4^+(D_2)}{dy} \Big) \\ &\times \Big\{ \rho_3(D_1) \frac{dY_{2b}^-(D_1)}{dy} Y_3^+(D_1) - \rho_2(D_1)\, Y_{2b}^-(D_1) \frac{dY_3^+(D_1)}{dy} \Big\} \Big] \end{aligned}$$

$$\times \Big[Y_{2a}^{+}(y_o)\Big(\rho_1(0)\,Y_1^{-}(0)\,\frac{dY_{2a}^{-}(0)}{dy} - \rho_2(0)\,\frac{dY_1^{-}(0)}{dy}\,Y_{2a}^{-}(0)\Big)$$

$$+ Y_{2a}^{-}(y_o)\Big(\rho_2(0)\,\frac{dY_1^{-}(0)}{dy}\,Y_{2a}^{+}(0) - \rho_1(0)\,Y_1^{-}(0)\,\frac{dY_{2a}^{+}(0)}{dy}\Big)\Big]\Big] \,/$$

$$\Big[\Big[\Big(\rho_3(D_2)\,Y_3^{+}(D_2)\,\frac{dY_4^{+}(D_2)}{dy} - \rho_4(D_2)\,\frac{dY_3^{+}(D_2)}{dy}\,Y_4^{+}(D_2)\Big)$$

$$\times \Big\{\rho_3(D_1)\,\frac{dY_{2b}^{+}(D_1)}{dy}\,Y_3^{-}(D_1) - \rho_2(D_1)\,Y_{2b}^{+}(D_1)\,\frac{dY_3^{-}(D_1)}{dy}\Big\}$$

$$+ \Big(\rho_4(D_2)\,\frac{dY_3^{-}(D_2)}{dy}\,Y_4^{+}(D_2) - \rho_3(D_2)\,Y_3^{-}(D_2)\,\frac{dY_4^{+}(D_2)}{dy}\Big)$$

$$\times \Big\{\rho_3(D_1)\,\frac{dY_{2b}^{+}(D_1)}{dy}\,Y_3^{+}(D_1) - \rho_2(D_1)\,Y_{2b}^{+}(D_1)\,\frac{dY_3^{+}(D_1)}{dy}\Big\}\Big]$$

$$\times \Big[\Big(Y_{2a}^{+}(y_o)\,\frac{dY_{2b}^{-}(y_o)}{dy} - \frac{dY_{2a}^{+}(y_o)}{dy}\,Y_{2b}^{-}(y_o)\Big)$$

$$\times \Big\{\rho_1(0)\,Y_1^{-}(0)\,\frac{dY_{2a}^{-}(0)}{dy} - \rho_2(0)\,\frac{dY_1^{-}(0)}{dy}\,Y_{2a}^{-}(0)\Big\}$$

$$+\left(\frac{dY^-_{2a}(y_o)}{dy}\,Y^-_{2b}(y_o) - Y^-_{2a}(y_o)\,\frac{dY^-_{2b}(y_o)}{dy}\right)$$

$$\times\left\{\rho_1(0)\,Y^-_1(0)\,\frac{dY^+_{2a}(0)}{dy} - \rho_2(0)\,\frac{dY^-_1(0)}{dy}\,Y^+_{2a}(0)\right\}\Big]$$

$$-\Big[\Big(\rho_3(D_2)\,Y^+_3(D_2)\,\frac{dY^+_4(D_2)}{dy} - \rho_4(D_2)\,\frac{dY^+_3(D_2)}{dy}\,Y^+_4(D_2)\Big)$$

$$\times\left\{\rho_3(D_1)\,\frac{dY^-_{2b}(D_1)}{dy}\,Y^-_3(D_1) - \rho_2(D_1)\,Y^-_{2b}(D_1)\,\frac{dY^-_3(D_1)}{dy}\right\}$$

$$+\left(\rho_4(D_2)\,\frac{dY^-_3(D_2)}{dy}\,Y^+_4(D_2) - \rho_3(D_2)\,Y^-_3(D_2)\,\frac{dY^+_4(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\,\frac{dY^-_{2b}(D_1)}{dy}\,Y^+_3(D_1) - \rho_2(D_1)\,Y^-_{2b}(D_1)\,\frac{dY^+_3(D_1)}{dy}\right\}\Big]$$

$$\times\Big[\Big(Y^+_{2a}(y_o)\,\frac{dY^+_{2b}(y_o)}{dy} - \frac{dY^+_{2a}(y_o)}{dy}\,Y^+_{2b}(y_o)\Big)$$

$$\times\left\{\rho_1(0)\,Y^-_1(0)\,\frac{dY^-_{2a}(0)}{dy} - \rho_2(0)\,\frac{dY^-_1(0)}{dy}\,Y^-_{2a}(0)\right\}$$

$$+\left(\frac{dY_{2a}^{-}(y_o)}{dy} Y_{2b}^{+}(y_o) - Y_{2a}^{-}(y_o) \frac{dY_{2b}^{+}(y_o)}{dy}\right)$$

$$\times \left\{\rho_1(0)\, Y_1^{-}(0) \frac{dY_{2a}^{+}(0)}{dy} - \rho_2(0) \frac{dY_1^{-}(0)}{dy} Y_{2a}^{+}(0)\right\}\Big]\Big]. \qquad (340)$$

For constant speed of sound and ambient density, (340) becomes

$$A_{2b_c} = \frac{-jk_f}{2\pi} e^{+jk_{y_2} D_1} e^{-jk_{y_4} D_2} \Big[\Big[\left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_3 k_{y_4} - \rho_4 k_{y_3}\right) e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}\Big]$$

$$\times \Big[\left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) e^{-jk_{y_2} y_o} + \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) e^{+jk_{y_2} y_o}\Big]\Big] \Big/$$

$$\Big[2 k_{y_2} e^{-jk_{y_4} D_2} \Big[\left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big]\Big] . \qquad (341)$$

Using the definitions of R_{21}, R_{23}, and R_{34} presented earlier, (341) may be reduced to

$$A_{2b_c} = \frac{+jk_r}{4\pi k_{y_2}} \left[\left[1 + R_{23} R_{34} e^{-j2k_{y_3}(D_2 - D_1)} \right] \right.$$

$$\left. \times \left[e^{+jk_{y_2} y_0} + R_{21} e^{-jk_{y_2} y_0} \right] \right] \Big/$$

$$\left[R_{23} R_{34} e^{-j2k_{y_3}(D_2 - D_1)} + 1 - R_{21} R_{34} e^{-j2k_{y_3}(D_2 - D_1)} - R_{21} R_{23} e^{-j2k_{y_2} D_1} \right].$$

$$(342)$$

Using conditions (328) and (329), (342) may be reduced to

$$A_{2b_c} = \frac{+jk_r}{4\pi k_{y_2}} \left[e^{+jk_{y_2} y_0} + R_{21} e^{-jk_{y_2} y_0} \right] \Big/$$

$$\left[1 - R_{21} R_{23} e^{-j2k_{y_2} D} \right]. \qquad (343)$$

Equation (343) is equal to equation (256). Since (256) has already been verified, we can conclude that our derivation of the solutions for $A_{2b'}$, and hence, for A_{2b_c} for the four media waveguide problem is correct.

For the unknown constant B_{2b}, *Mathematica* provided the following numerator:

$$
\begin{aligned}
num_5 = G_1 \Big[& a_{1,3}\, a_{2,2}\, a_{3,7}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,6} \\
& - a_{1,2}\, a_{2,3}\, a_{3,7}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,6} + a_{1,1}\, a_{2,3}\, a_{3,7}\, a_{4,8}\, a_{5,2}\, a_{6,4}\, a_{7,6} \\
& - a_{1,1}\, a_{2,2}\, a_{3,7}\, a_{4,8}\, a_{5,3}\, a_{6,4}\, a_{7,6} - a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,7}\, a_{7,6} \\
& + a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,7}\, a_{7,6} - a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,8}\, a_{5,2}\, a_{6,7}\, a_{7,6} \\
& + a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,8}\, a_{5,3}\, a_{6,7}\, a_{7,6} - a_{1,3}\, a_{2,2}\, a_{3,6}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,7} \\
& + a_{1,2}\, a_{2,3}\, a_{3,6}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,7} - a_{1,1}\, a_{2,3}\, a_{3,6}\, a_{4,8}\, a_{5,2}\, a_{6,4}\, a_{7,7} \\
& + a_{1,1}\, a_{2,2}\, a_{3,6}\, a_{4,8}\, a_{5,3}\, a_{6,4}\, a_{7,7} + a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,6}\, a_{7,7} \\
& - a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,6}\, a_{7,7} + a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,8}\, a_{5,2}\, a_{6,6}\, a_{7,7} \\
& - a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,8}\, a_{5,3}\, a_{6,6}\, a_{7,7} - a_{1,3}\, a_{2,2}\, a_{3,7}\, a_{4,6}\, a_{5,1}\, a_{6,4}\, a_{7,8} \\
& + a_{1,2}\, a_{2,3}\, a_{3,7}\, a_{4,6}\, a_{5,1}\, a_{6,4}\, a_{7,8} + a_{1,3}\, a_{2,2}\, a_{3,6}\, a_{4,7}\, a_{5,1}\, a_{6,4}\, a_{7,8} \\
& - a_{1,2}\, a_{2,3}\, a_{3,6}\, a_{4,7}\, a_{5,1}\, a_{6,4}\, a_{7,8} - a_{1,1}\, a_{2,3}\, a_{3,7}\, a_{4,6}\, a_{5,2}\, a_{6,4}\, a_{7,8} \\
& + a_{1,1}\, a_{2,3}\, a_{3,6}\, a_{4,7}\, a_{5,2}\, a_{6,4}\, a_{7,8} + a_{1,1}\, a_{2,2}\, a_{3,7}\, a_{4,6}\, a_{5,3}\, a_{6,4}\, a_{7,8} \\
& - a_{1,1}\, a_{2,2}\, a_{3,6}\, a_{4,7}\, a_{5,3}\, a_{6,4}\, a_{7,8} - a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,7}\, a_{5,1}\, a_{6,6}\, a_{7,8} \\
& + a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,7}\, a_{5,1}\, a_{6,6}\, a_{7,8} - a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,7}\, a_{5,2}\, a_{6,6}\, a_{7,8} \\
& + a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,7}\, a_{5,3}\, a_{6,6}\, a_{7,8} + a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,6}\, a_{5,1}\, a_{6,7}\, a_{7,8} \\
& - a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,6}\, a_{5,1}\, a_{6,7}\, a_{7,8} + a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,6}\, a_{5,2}\, a_{6,7}\, a_{7,8} \\
& - a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,6}\, a_{5,3}\, a_{6,7}\, a_{7,8} \Big] .
\end{aligned} \tag{344}
$$

Factoring (344) and collecting common terms yields the following *generic expression* for the *numerator* of B_{2b}:

$$
\begin{aligned}
num_5 = G_1 \Big[& \left(a_{4,8}\, a_{7,6} - a_{4,6}\, a_{7,8}\right) \left\{a_{3,4}\, a_{6,7} - a_{3,7}\, a_{6,4}\right\} \\
& + \left(a_{4,7}\, a_{7,8} - a_{4,8}\, a_{7,7}\right) \left\{a_{3,4}\, a_{6,6} - a_{3,6}\, a_{6,4}\right\} \Big] \\
& \times \left[a_{2,2} \left(a_{1,1}\, a_{5,3} - a_{1,3}\, a_{5,1}\right) + a_{2,3} \left(a_{1,2}\, a_{5,1} - a_{1,1}\, a_{5,2}\right)\right] .
\end{aligned} \tag{345}
$$

Substituting appropriate expressions into (345) and using (315) yields the following *general expression* for B_{2b}:

$$B_{2b} = \frac{-k_r}{2\pi}\Bigg[\Bigg(\rho_3(D_2)\,Y_3^+(D_2)\,\frac{dY_4^+(D_2)}{dy} - \rho_4(D_2)\,\frac{dY_3^+(D_2)}{dy}\,Y_4^+(D_2)\Bigg)$$

$$\times\left\{\rho_3(D_1)\,\frac{dY_{2b}^+(D_1)}{dy}\,Y_3^-(D_1) - \rho_2(D_1)\,Y_{2b}^+(D_1)\,\frac{dY_3^-(D_1)}{dy}\right\}$$

$$+\Bigg(\rho_4(D_2)\,\frac{dY_3^-(D_2)}{dy}\,Y_4^+(D_2) - \rho_3(D_2)\,Y_3^-(D_2)\,\frac{dY_4^+(D_2)}{dy}\Bigg)$$

$$\times\left\{\rho_3(D_1)\,\frac{dY_{2b}^+(D_1)}{dy}\,Y_3^+(D_1) - \rho_2(D_1)\,Y_{2b}^+(D_1)\,\frac{dY_3^+(D_1)}{dy}\right\}\Bigg]$$

$$\times\Bigg[Y_{2a}^+(y_o)\Bigg(\rho_2(0)\,\frac{dY_1^-(0)}{dy}\,Y_{2a}^-(0) - \rho_1(0)\,Y_1^-(0)\,\frac{dY_{2a}^-(0)}{dy}\Bigg)$$

$$+\,Y_{2a}^-(y_o)\Bigg(\rho_1(0)\,Y_1^-(0)\,\frac{dY_{2a}^+(0)}{dy} - \rho_2(0)\,\frac{dY_1^-(0)}{dy}\,Y_{2a}^+(0)\Bigg)\Bigg]\Bigg] \Big/$$

$$\Bigg[\Bigg[\Bigg(\rho_3(D_2)\,Y_3^+(D_2)\,\frac{dY_4^+(D_2)}{dy} - \rho_4(D_2)\,\frac{dY_3^+(D_2)}{dy}\,Y_4^+(D_2)\Bigg)$$

$$\times\left\{\rho_3(D_1)\,\frac{dY_{2b}^+(D_1)}{dy}\,Y_3^-(D_1) - \rho_2(D_1)\,Y_{2b}^+(D_1)\,\frac{dY_3^-(D_1)}{dy}\right\}$$

$$+\left(\rho_4(D_2)\frac{dY_3^-(D_2)}{dy}Y_4^+(D_2)-\rho_3(D_2)Y_3^-(D_2)\frac{dY_4^+(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^+(D_1)}{dy}Y_3^+(D_1)-\rho_2(D_1)Y_{2b}^+(D_1)\frac{dY_3^+(D_1)}{dy}\right\}\Big]$$

$$\times\Big[\left(Y_{2a}^+(y_o)\frac{dY_{2b}^-(y_o)}{dy}-\frac{dY_{2a}^+(y_o)}{dy}Y_{2b}^-(y_o)\right)$$

$$\times\left\{\rho_1(0)Y_1^-(0)\frac{dY_{2a}^-(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^-(0)\right\}$$

$$+\left(\frac{dY_{2a}^-(y_o)}{dy}Y_{2b}^-(y_o)-Y_{2a}^-(y_o)\frac{dY_{2b}^-(y_o)}{dy}\right)$$

$$\times\left\{\rho_1(0)Y_1^-(0)\frac{dY_{2a}^+(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^+(0)\right\}\Big]$$

$$-\Big[\left(\rho_3(D_2)Y_3^+(D_2)\frac{dY_4^+(D_2)}{dy}-\rho_4(D_2)\frac{dY_3^+(D_2)}{dy}Y_4^+(D_2)\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^-(D_1)}{dy}Y_3^-(D_1)-\rho_2(D_1)Y_{2b}^-(D_1)\frac{dY_3^-(D_1)}{dy}\right\}$$

$$+\left(\rho_4(D_2)\frac{dY_3^-(D_2)}{dy}Y_4^+(D_2)-\rho_3(D_2)\,Y_3^-(D_2)\frac{dY_4^+(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^-(D_1)}{dy}Y_3^+(D_1)-\rho_2(D_1)\,Y_{2b}^-(D_1)\frac{dY_3^+(D_1)}{dy}\right\}\Big]$$

$$\times\left[\left(Y_{2a}^+(y_o)\frac{dY_{2b}^+(y_o)}{dy}-\frac{dY_{2a}^+(y_o)}{dy}Y_{2b}^+(y_o)\right)\right.$$

$$\times\left\{\rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^-(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^-(0)\right\}$$

$$+\left(\frac{dY_{2a}^-(y_o)}{dy}Y_{2b}^+(y_o)-Y_{2a}^-(y_o)\frac{dY_{2b}^+(y_o)}{dy}\right)$$

$$\times\left\{\rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^+(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^+(0)\right\}\Big]\Big]\,. \qquad (346)$$

For constant speed of sound and ambient density, (346) becomes

$$B_{2b_c}=\frac{+jk_r}{2\pi}\,e^{-jk_{y_2}D_1}\,e^{-jk_{y_4}D_2}\Big[\Big[\left(\rho_3\,k_{y_2}+\rho_2\,k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right) \left(\rho_3 k_{y_4} + \rho_4 k_{y_3}\right) e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big]$$

$$\times \Big[\left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) e^{-jk_{y_2} y_0} + \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) e^{+jk_{y_2} y_0}\Big]\Big] \Big/$$

$$\Big[2 k_{y_2} e^{-jk_{y_4} D_2} \Big[\left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_1 k_{y_2} + \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{+jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big]\Big] . \qquad (347)$$

Using the definitions of R_{21}, R_{23}, and R_{34} presented earlier, (347) may be reduced to

$$B_{2b_c} = \frac{+jk_r}{4\pi k_{y_2}} \left[R_{23} + R_{34}\, e^{-j2k_{y_3}(D_2 - D_1)}\right]$$

$$\times \left[1 + R_{21}\, e^{-j2k_{y_2} y_o}\right] e^{+jk_{y_2} y_o}\, e^{-j2k_{y_2} D_1} \;/$$

$$\Big[R_{23} R_{34} e^{-j2k_{y_3}(D_2 - D_1)} + 1$$

$$- R_{21} R_{34} e^{-j2k_{y_2} D_1} e^{-j2k_{y_3}(D_2 - D_1)} - R_{21} R_{23} e^{-j2k_{y_2} D_1}\Big]. \qquad (348)$$

Using conditions (328) and (329), (348) reduces to

$$B_{2b_c} = \frac{+jk_r}{4\pi k_{y_2}} R_{23} \left[e^{+jk_{y_2} y_0} + R_{21} e^{-jk_{y_2} y_0}\right] e^{-j2k_{y_2} D} \Big/$$

$$\left[1 - R_{21} R_{23} e^{-j2k_{y_2} D}\right]. \qquad (349)$$

Equation (349) is equivalent to equation (261). Since (261) has already been verified, we can conclude that our derivation of the solutions for B_{2b}, and hence, for B_{2b_c} for the four media waveguide problem is correct.

For the unknown constant A_3, *Mathematica* provided the following numerator:

$$\begin{aligned} num_6 = G_1 \big[& a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,7} \\ & - a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,7} + a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,8}\, a_{5,2}\, a_{6,4}\, a_{7,7} \\ & - a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,8}\, a_{5,3}\, a_{6,4}\, a_{7,7} - a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,7} \\ & + a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,7} - a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,8}\, a_{5,2}\, a_{6,5}\, a_{7,7} \\ & + a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,8}\, a_{5,3}\, a_{6,5}\, a_{7,7} - a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,7}\, a_{5,1}\, a_{6,4}\, a_{7,8} \end{aligned}$$

$$+ a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,7}\, a_{5,1}\, a_{6,4}\, a_{7,8} - a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,7}\, a_{5,2}\, a_{6,4}\, a_{7,8}$$
$$+ a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,7}\, a_{5,3}\, a_{6,4}\, a_{7,8} + a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,7}\, a_{5,1}\, a_{6,5}\, a_{7,8}$$
$$- a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,7}\, a_{5,1}\, a_{6,5}\, a_{7,8} + a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,7}\, a_{5,2}\, a_{6,5}\, a_{7,8}$$
$$- a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,7}\, a_{5,3}\, a_{6,5}\, a_{7,8} \Big] . \tag{350}$$

Factoring (350) and collecting common terms yields the following *generic expression* for the *numerator* of A_3:

$$\begin{aligned} num_6 = G_1 \left(a_{4,8}\, a_{7,7} - a_{4,7}\, a_{7,8}\right) \left\{a_{3,5}\, a_{6,4} - a_{3,4}\, a_{6,5}\right\} \\ \times \left[a_{2,2} \left(a_{1,3}\, a_{5,1} - a_{1,1}\, a_{5,3}\right) + a_{2,3} \left(a_{1,1}\, a_{5,2} - a_{1,2}\, a_{5,1}\right)\right] . \end{aligned} \tag{351}$$

Substituting appropriate expressions into (351) and using (315) yields the following *general expression* for A_3:

$$A_3 = \frac{-k_r}{2\pi} \left(\rho_3(D_2)\, Y_3^-(D_2) \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2) \frac{dY_3^-(D_2)}{dy} Y_4^+(D_2)\right)$$

$$\times \left\{\rho_2(D_1) \frac{dY_{2b}^+(D_1)}{dy} Y_{2b}^-(D_1) - \rho_2(D_1)\, Y_{2b}^+(D_1) \frac{dY_{2b}^-(D_1)}{dy}\right\}$$

$$\times \Bigg[Y_{2a}^+(y_o) \left(\rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^-(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^-(0)\right) + Y_{2a}^-(y_o)$$

$$\times \left(\rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(0) - \rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^+(0)}{dy}\right)\Bigg] \Big/$$

$$\Big[\Big(\rho_3(D_2)\,Y_3^+(D_2)\,\frac{dY_4^+(D_2)}{dy} - \rho_4(D_2)\,\frac{dY_3^+(D_2)}{dy}\,Y_4^+(D_2)\Big)$$

$$\times\Big\{\rho_3(D_1)\,\frac{dY_{2b}^+(D_1)}{dy}\,Y_3^-(D_1) - \rho_2(D_1)\,Y_{2b}^+(D_1)\,\frac{dY_3^-(D_1)}{dy}\Big\}$$

$$+\Big(\rho_4(D_2)\,\frac{dY_3^-(D_2)}{dy}\,Y_4^+(D_2) - \rho_3(D_2)\,Y_3^-(D_2)\,\frac{dY_4^+(D_2)}{dy}\Big)$$

$$\times\Big\{\rho_3(D_1)\,\frac{dY_{2b}^+(D_1)}{dy}\,Y_3^+(D_1) - \rho_2(D_1)\,Y_{2b}^+(D_1)\,\frac{dY_3^+(D_1)}{dy}\Big\}\Big]$$

$$\times\Big[\Big(Y_{2a}^+(y_o)\,\frac{dY_{2b}^-(y_o)}{dy} - \frac{dY_{2a}^+(y_o)}{dy}\,Y_{2b}^-(y_o)\Big)$$

$$\times\Big\{\rho_1(0)\,Y_1^-(0)\,\frac{dY_{2a}^-(0)}{dy} - \rho_2(0)\,\frac{dY_1^-(0)}{dy}\,Y_{2a}^-(0)\Big\}$$

$$+\Big(\frac{dY_{2a}^-(y_o)}{dy}\,Y_{2b}^-(y_o) - Y_{2a}^-(y_o)\,\frac{dY_{2b}^-(y_o)}{dy}\Big)$$

$$\times\Big\{\rho_1(0)\,Y_1^-(0)\,\frac{dY_{2a}^+(0)}{dy} - \rho_2(0)\,\frac{dY_1^-(0)}{dy}\,Y_{2a}^+(0)\Big\}\Big]$$

$$- \left[\left(\rho_3(D_2)\, Y_3^+(D_2) \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2) \frac{dY_3^+(D_2)}{dy} Y_4^+(D_2) \right) \right.$$

$$\times \left\{ \rho_3(D_1) \frac{dY_{2b}^-(D_1)}{dy} Y_3^-(D_1) - \rho_2(D_1)\, Y_{2b}^-(D_1) \frac{dY_3^-(D_1)}{dy} \right\}$$

$$+ \left(\rho_4(D_2) \frac{dY_3^-(D_2)}{dy} Y_4^+(D_2) - \rho_3(D_2)\, Y_3^-(D_2) \frac{dY_4^+(D_2)}{dy} \right)$$

$$\left. \times \left\{ \rho_3(D_1) \frac{dY_{2b}^-(D_1)}{dy} Y_3^+(D_1) - \rho_2(D_1)\, Y_{2b}^-(D_1) \frac{dY_3^+(D_1)}{dy} \right\} \right]$$

$$\times \left[\left(Y_{2a}^+(y_0) \frac{dY_{2b}^+(y_0)}{dy} - \frac{dY_{2a}^+(y_0)}{dy} Y_{2b}^+(y_0) \right) \right.$$

$$\times \left\{ \rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^-(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^-(0) \right\}$$

$$+ \left(\frac{dY_{2a}^-(y_0)}{dy} Y_{2b}^+(y_0) - Y_{2a}^-(y_0) \frac{dY_{2b}^+(y_0)}{dy} \right)$$

$$\left. \times \left\{ \rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^+(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^+(0) \right\} \right] \right] . \qquad (352)$$

For constant speed of sound and ambient density, (352) becomes

$$A_{3_c} = \frac{+jk_f}{2\pi} e^{+jk_{y_3} D_2} e^{-jk_{y_4} D_2} \Big[\big(2 \rho_2 k_{y_2}\big) \big(\rho_4 k_{y_3} + \rho_3 k_{y_4}\big)$$

$$\times \Big[\big(\rho_1 k_{y_2} - \rho_2 k_{y_1}\big) e^{-jk_{y_2} y_0} + \big(\rho_1 k_{y_2} + \rho_2 k_{y_1}\big) e^{+jk_{y_2} y_0} \Big]\Big] \Big/$$

$$\Big[2 k_{y_2} e^{-jk_{y_4} D_2} \Big[\big(\rho_1 k_{y_2} + \rho_2 k_{y_1}\big) \big(\rho_3 k_{y_2} - \rho_2 k_{y_3}\big)$$

$$\times \big(\rho_4 k_{y_3} - \rho_3 k_{y_4}\big) e^{+jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \big(\rho_1 k_{y_2} + \rho_2 k_{y_1}\big) \big(\rho_3 k_{y_2} + \rho_2 k_{y_3}\big)$$

$$\times \big(\rho_4 k_{y_3} + \rho_3 k_{y_4}\big) e^{+jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$- \big(\rho_1 k_{y_2} - \rho_2 k_{y_1}\big) \big(\rho_3 k_{y_2} + \rho_2 k_{y_3}\big)$$

$$\times\left(\rho_4 k_{y_3} - \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1}\, e^{-jk_{y_3} D_2}\, e^{+jk_{y_3} D_1}$$

$$-\left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times\left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1}\, e^{+jk_{y_3} D_2}\, e^{-jk_{y_3} D_1}\Big]\Big]. \qquad (353)$$

Using the definitions of R_{21}, R_{23}, R_{34}, and T_{23} presented earlier, (353) may be reduced to

$$A_{3_c} = \frac{+jk_r}{4\pi k_{y_2}} T_{23}\left[e^{+jk_{y_2} y_o} + R_{21}\, e^{-jk_{y_2} y_o}\right] e^{-jk_{y_2} D_1}\, e^{+jk_{y_3} D_2} \Big/$$

$$\Big[R_{23} R_{34}\, e^{-jk_{y_3}(D_2 - D_1)} + e^{+jk_{y_3}(D_2 - D_1)} - R_{21} R_{34}\, e^{-jk_{y_3}(D_2 - D_1)}\, e^{-j2k_{y_2} D_1}$$

$$- R_{21} R_{23}\, e^{+k_{y_3}(D_2 - D_1)}\, e^{-j2k_{y_2} D_1}\Big]. \qquad (354)$$

Using conditions (328), (329), (354) may be reduced to

$$A_{3_c} = \frac{+jk_r}{4\pi k_{y_2}} T_{23} \left[e^{+jk_{y_2} y_o} + R_{21} e^{-jk_{y_2} y_o} \right] e^{-jk_{y_2} D} \; e^{+jk_{y_3} D} \; / $$

$$\left[1 - R_{21} R_{23} e^{-j2k_{y_2} D} \right]. \qquad (355)$$

Equation (355) is equal to equation (274). Since (274) has already been verified, we can conclude that our derivation of the solutions for A_3, and hence, for A_{3_c} for the four media waveguide problem is correct.

For the unknown constant B_3, *Mathematica* provided the following numerator:

$$\begin{aligned} num_7 = G_1 \Big[& - a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,6} \\ & + a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,8}\, a_{5,1}\, a_{6,4}\, a_{7,6} - a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,8}\, a_{5,2}\, a_{6,4}\, a_{7,6} \\ & + a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,8}\, a_{5,3}\, a_{6,4}\, a_{7,6} + a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,6} \\ & - a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,8}\, a_{5,1}\, a_{6,5}\, a_{7,6} + a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,8}\, a_{5,2}\, a_{6,5}\, a_{7,6} \\ & - a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,8}\, a_{5,3}\, a_{6,5}\, a_{7,6} + a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,6}\, a_{5,1}\, a_{6,4}\, a_{7,8} \\ & - a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,6}\, a_{5,1}\, a_{6,4}\, a_{7,8} + a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,6}\, a_{5,2}\, a_{6,4}\, a_{7,8} \\ & - a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,6}\, a_{5,3}\, a_{6,4}\, a_{7,8} - a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,6}\, a_{5,1}\, a_{6,5}\, a_{7,8} \\ & + a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,6}\, a_{5,1}\, a_{6,5}\, a_{7,8} - a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,6}\, a_{5,2}\, a_{6,5}\, a_{7,8} \\ & + a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,6}\, a_{5,3}\, a_{6,5}\, a_{7,8} \Big]. \end{aligned} \qquad (356)$$

Factoring (356) and collecting common terms yields the following *generic expression* for the *numerator* of B_3:

$$\mathrm{num}_7 = G_1\left(a_{4,6}\,a_{7,8} - a_{4,8}\,a_{7,6}\right)\left\{a_{3,5}\,a_{6,4} - a_{3,4}\,a_{6,5}\right\}$$
$$\times\left[a_{2,2}\left(a_{1,3}\,a_{5,1} - a_{1,1}\,a_{5,3}\right) + a_{2,3}\left(a_{1,1}\,a_{5,2} - a_{1,2}\,a_{5,1}\right)\right]. \qquad (357)$$

Substituting appropriate expressions into (357) and using (315) yields the following *general expression* for B_3:

$$B_3 = \frac{-k_r}{2\pi}\left(\rho_4(D_2)\,\frac{dY_3^+(D_2)}{dy}\,Y_4^+(D_2) - \rho_3(D_2)\,Y_3^+(D_2)\,\frac{dY_4^+(D_2)}{dy}\right)$$

$$\times\left\{\rho_2(D_1)\,\frac{dY_{2b}^+(D_1)}{dy}\,Y_{2b}^-(D_1) - \rho_2(D_1)\,Y_{2b}^+(D_1)\,\frac{dY_{2b}^-(D_1)}{dy}\right\}$$

$$\times\left[Y_{2a}^+(y_o)\left(\rho_1(0)\,Y_1^-(0)\,\frac{dY_{2a}^-(0)}{dy} - \rho_2(0)\,\frac{dY_1^-(0)}{dy}\,Y_{2a}^-(0)\right) + Y_{2a}^-(y_o)\right.$$

$$\left.\times\left(\rho_2(0)\,\frac{dY_1^-(0)}{dy}\,Y_{2a}^+(0) - \rho_1(0)\,Y_1^-(0)\,\frac{dY_{2a}^+(0)}{dy}\right)\right] \Big/$$

$$\left[\left[\left(\rho_3(D_2)\,Y_3^+(D_2)\,\frac{dY_4^+(D_2)}{dy} - \rho_4(D_2)\,\frac{dY_3^+(D_2)}{dy}\,Y_4^+(D_2)\right)\right.\right.$$

$$\times\left\{\rho_3(D_1)\,\frac{dY_{2b}^+(D_1)}{dy}\,Y_3^-(D_1) - \rho_2(D_1)\,Y_{2b}^+(D_1)\,\frac{dY_3^-(D_1)}{dy}\right\}$$

$$+\left(\rho_4(D_2)\frac{dY_3^-(D_2)}{dy}Y_4^+(D_2)-\rho_3(D_2)\,Y_3^-(D_2)\frac{dY_4^+(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^+(D_1)}{dy}Y_3^+(D_1)-\rho_2(D_1)\,Y_{2b}^+(D_1)\frac{dY_3^+(D_1)}{dy}\right\}\Big]$$

$$\times\Big[\left(Y_{2a}^+(y_o)\frac{dY_{2b}^-(y_o)}{dy}-\frac{dY_{2a}^+(y_o)}{dy}Y_{2b}^-(y_o)\right)$$

$$\times\left\{\rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^-(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^-(0)\right\}$$

$$+\left(\frac{dY_{2a}^-(y_o)}{dy}Y_{2b}^-(y_o)-Y_{2a}^-(y_o)\frac{dY_{2b}^-(y_o)}{dy}\right)$$

$$\times\left\{\rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^+(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^+(0)\right\}\Big]$$

$$-\Big[\left(\rho_3(D_2)\,Y_3^+(D_2)\frac{dY_4^+(D_2)}{dy}-\rho_4(D_2)\frac{dY_3^+(D_2)}{dy}Y_4^+(D_2)\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^-(D_1)}{dy}Y_3^-(D_1)-\rho_2(D_1)\,Y_{2b}^-(D_1)\frac{dY_3^-(D_1)}{dy}\right\}$$

$$+\left(\rho_4(D_2)\frac{dY_3^-(D_2)}{dy}Y_4^+(D_2)-\rho_3(D_2)Y_3^-(D_2)\frac{dY_4^+(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^-(D_1)}{dy}Y_3^+(D_1)-\rho_2(D_1)Y_{2b}^-(D_1)\frac{dY_3^+(D_1)}{dy}\right\}\Big]$$

$$\times\Big[\left(Y_{2a}^+(y_o)\frac{dY_{2b}^+(y_o)}{dy}-\frac{dY_{2a}^+(y_o)}{dy}Y_{2b}^+(y_o)\right)$$

$$\times\left\{\rho_1(0)Y_1^-(0)\frac{dY_{2a}^-(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^-(0)\right\}$$

$$\left(\frac{dY_{2a}^-(y_o)}{dy}Y_{2b}^+(y_o)-Y_{2a}^-(y_o)\frac{dY_{2b}^+(y_o)}{dy}\right)$$

$$\times\left\{\rho_1(0)Y_1^-(0)\frac{dY_{2a}^+(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^+(0)\right\}\Big]\Big]\,. \qquad (358)$$

For constant speed of sound and ambient density, (358) becomes

$$B_{3_c}=\frac{+jk_r}{2\pi}e^{-jk_{y_3}D_2}e^{-jk_{y_4}D_2}\left(2\rho_2 k_{y_2}\right)\left(\rho_4 k_{y_3}-\rho_3 k_{y_4}\right)$$

$$\times \left[\left(\rho_1 k_{y_2} - \rho_2 k_{y_1} \right) e^{-jk_{y_2} y_0} + \left(\rho_1 k_{y_2} + \rho_2 k_{y_1} \right) e^{+jk_{y_2} y_0} \right] /$$

$$\left[2 k_{y_2} e^{-jk_{y_4} D_2} \left[\left(\rho_1 k_{y_2} + \rho_2 k_{y_1} \right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3} \right) \right.\right.$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4} \right) e^{+jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_1 k_{y_2} + \rho_2 k_{y_1} \right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3} \right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4} \right) e^{+jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1} \right) \left(\rho_3 k_{y_2} + \rho_2 k_{y_3} \right)$$

$$\times \left(\rho_4 k_{y_3} - \rho_3 k_{y_4} \right) e^{-jk_{y_2} D_1} e^{-jk_{y_3} D_2} e^{+jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1} \right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3} \right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big] . \qquad (359)$$

Using the definitions of R_{21}, R_{23}, R_{34}, and T_{23} presented earlier, (359) may be reduced to

$$B_{3_c} = \frac{+jk_r}{4\pi k_{y_2}} R_{34} T_{23} \left[e^{+jk_{y_2} y_o} + R_{21} e^{-jk_{y_2} y_o}\right] e^{-jk_{y_2} D_1} e^{-jk_{y_3} D_2} \Big/$$

$$\Big[R_{23} R_{34} e^{-jk_{y_3}(D_2 - D_1)} + e^{+jk_{y_3}(D_2 - D_1)} - R_{21} R_{34} e^{-jk_{y_3}(D_2 - D_1)} e^{-j2k_{y_2} D_1}$$

$$- R_{21} R_{23} e^{+jk_{y_3}(D_2 - D_1)} e^{-j2k_{y_2} D_1} \Big] . \qquad (360)$$

Using conditions (328), (329), (360) may be reduced to

$$B_{3_c} = 0 . \qquad (361)$$

Equation (361) is exactly what one would expect for a semi-infinite medium (i.e., no wave propagation in the negative y direction). Thus, we can

conclude that our derivation of the solutions for B_3, and hence, for B_{3_c} for the four media waveguide problem is correct.

For the unknown constant B_1, *Mathematica* provided the following numerator:

$$
\begin{aligned}
\mathrm{num}_1 = G_1 \Big[& a_{1,3}\, a_{2,5}\, a_{3,7}\, a_{4,8}\, a_{5,2}\, a_{6,4}\, a_{7,6} \\
& - a_{1,2}\, a_{2,5}\, a_{3,7}\, a_{4,8}\, a_{5,3}\, a_{6,4}\, a_{7,6} - a_{1,3}\, a_{2,4}\, a_{3,7}\, a_{4,8}\, a_{5,2}\, a_{6,5}\, a_{7,6} \\
& + a_{1,2}\, a_{2,4}\, a_{3,7}\, a_{4,8}\, a_{5,3}\, a_{6,5}\, a_{7,6} - a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,2}\, a_{6,7}\, a_{7,6} \\
& + a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,2}\, a_{6,7}\, a_{7,6} + a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,3}\, a_{6,7}\, a_{7,6} \\
& - a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,3}\, a_{6,7}\, a_{7,6} - a_{1,3}\, a_{2,5}\, a_{3,6}\, a_{4,8}\, a_{5,2}\, a_{6,4}\, a_{7,7} \\
& + a_{1,2}\, a_{2,5}\, a_{3,6}\, a_{4,8}\, a_{5,3}\, a_{6,4}\, a_{7,7} + a_{1,3}\, a_{2,4}\, a_{3,6}\, a_{4,8}\, a_{5,2}\, a_{6,5}\, a_{7,7} \\
& - a_{1,2}\, a_{2,4}\, a_{3,6}\, a_{4,8}\, a_{5,3}\, a_{6,5}\, a_{7,7} + a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,2}\, a_{6,6}\, a_{7,7} \\
& - a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,2}\, a_{6,6}\, a_{7,7} - a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,8}\, a_{5,3}\, a_{6,6}\, a_{7,7} \\
& + a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,8}\, a_{5,3}\, a_{6,6}\, a_{7,7} - a_{1,3}\, a_{2,5}\, a_{3,7}\, a_{4,6}\, a_{5,2}\, a_{6,4}\, a_{7,8} \\
& + a_{1,3}\, a_{2,5}\, a_{3,6}\, a_{4,7}\, a_{5,2}\, a_{6,4}\, a_{7,8} + a_{1,2}\, a_{2,5}\, a_{3,7}\, a_{4,6}\, a_{5,3}\, a_{6,4}\, a_{7,8} \\
& - a_{1,2}\, a_{2,5}\, a_{3,6}\, a_{4,7}\, a_{5,3}\, a_{6,4}\, a_{7,8} + a_{1,3}\, a_{2,4}\, a_{3,7}\, a_{4,6}\, a_{5,2}\, a_{6,5}\, a_{7,8} \\
& - a_{1,3}\, a_{2,4}\, a_{3,6}\, a_{4,7}\, a_{5,2}\, a_{6,5}\, a_{7,8} - a_{1,2}\, a_{2,4}\, a_{3,7}\, a_{4,6}\, a_{5,3}\, a_{6,5}\, a_{7,8} \\
& + a_{1,2}\, a_{2,4}\, a_{3,6}\, a_{4,7}\, a_{5,3}\, a_{6,5}\, a_{7,8} - a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,7}\, a_{5,2}\, a_{6,6}\, a_{7,8} \\
& + a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,7}\, a_{5,2}\, a_{6,6}\, a_{7,8} + a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,7}\, a_{5,3}\, a_{6,6}\, a_{7,8} \\
& - a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,7}\, a_{5,3}\, a_{6,6}\, a_{7,8} + a_{1,3}\, a_{2,5}\, a_{3,4}\, a_{4,6}\, a_{5,2}\, a_{6,7}\, a_{7,8} \\
& - a_{1,3}\, a_{2,4}\, a_{3,5}\, a_{4,6}\, a_{5,2}\, a_{6,7}\, a_{7,8} - a_{1,2}\, a_{2,5}\, a_{3,4}\, a_{4,6}\, a_{5,3}\, a_{6,7}\, a_{7,8} \\
& + a_{1,2}\, a_{2,4}\, a_{3,5}\, a_{4,6}\, a_{5,3}\, a_{6,7}\, a_{7,8} \Big] .
\end{aligned}
\tag{362}
$$

Factoring (362) and collecting common terms yields the following *generic expression* for the *numerator* of B_1:

$$
\begin{aligned}
\mathrm{num}_1 = G_1 \left(a_{1,3}\, a_{5,2} - a_{1,2}\, a_{5,3} \right) \Big[& a_{2,5} \Big[\left(a_{4,6}\, a_{7,8} - a_{4,8}\, a_{7,6} \right) \\
\times \left\{ a_{3,4}\, a_{6,7} - a_{3,7}\, a_{6,4} \right\} & + \left(a_{4,8}\, a_{7,7} - a_{4,7}\, a_{7,8} \right) \left\{ a_{3,4}\, a_{6,6} - a_{3,6}\, a_{6,4} \right\} \Big] \\
& - a_{2,4} \Big[\left(a_{4,6}\, a_{7,8} - a_{4,8}\, a_{7,6} \right) \left\{ a_{3,5}\, a_{6,7} - a_{3,7}\, a_{6,5} \right\} \\
& + \left(a_{4,8}\, a_{7,7} - a_{4,7}\, a_{7,8} \right) \left\{ a_{3,5}\, a_{6,6} - a_{3,6}\, a_{6,5} \right\} \Big] \Big] .
\end{aligned}
\tag{363}
$$

Substituting appropriate expressions into (363) and using (315) yields the following *general expression* for B_1:

$$B_1 = \frac{k_r}{2\pi}\left(\rho_2(0)\,\frac{dY^+_{2a}(0)}{dy}\,Y^-_{2a}(0) - \rho_2(0)\,Y^+_{2a}(0)\,\frac{dY^-_{2a}(0)}{dy}\right)$$

$$\times\Bigg[Y^-_{2b}(y_o)\,\Big[\left(\rho_4(D_2)\,\frac{dY^+_3(D_2)}{dy}\,Y^+_4(D_2) - \rho_3(D_2)\,Y^+_3(D_2)\,\frac{dY^+_4(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\,\frac{dY^+_{2b}(D_1)}{dy}\,Y^-_3(D_1) - \rho_2(D_1)\,Y^+_{2b}(D_1)\,\frac{dY^-_3(D_1)}{dy}\right\}$$

$$+\left(\rho_3(D_2)\,Y^-_3(D_2)\,\frac{dY^+_4(D_2)}{dy} - \rho_4(D_2)\,\frac{dY^-_3(D_2)}{dy}\,Y^+_4(D_2)\right)$$

$$\times\left\{\rho_3(D_1)\,\frac{dY^+_{2b}(D_1)}{dy}\,Y^+_3(D_1) - \rho_2(D_1)\,Y^+_{2b}(D_1)\,\frac{dY^+_3(D_1)}{dy}\right\}\Big]$$

$$- Y^+_{2b}(y_o)\,\Big[\left(\rho_4(D_2)\,\frac{dY^+_3(D_2)}{dy}\,Y^+_4(D_2) - \rho_3(D_2)\,Y^+_3(D_2)\,\frac{dY^+_4(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\,\frac{dY^-_{2b}(D_1)}{dy}\,Y^-_3(D_1) - \rho_2(D_1)\,Y^-_{2b}(D_1)\,\frac{dY^-_3(D_1)}{dy}\right\}$$

$$+ \left(\rho_3(D_2)\, Y_3^-(D_2) \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2) \frac{dY_3^-(D_2)}{dy} Y_4^+(D_2)\right)$$

$$\times \left\{\rho_3(D_1) \frac{dY_{2b}^-(D_1)}{dy} Y_3^+(D_1) - \rho_2(D_1)\, Y_{2b}^-(D_1) \frac{dY_3^+(D_1)}{dy}\right\}\Big]\Big] \Big/$$

$$\Big[\Big[\left(\rho_3(D_2)\, Y_3^+(D_2) \frac{dY_4^+(D_2)}{dy} - \rho_4(D_2) \frac{dY_3^+(D_2)}{dy} Y_4^+(D_2)\right)$$

$$\times \left\{\rho_3(D_1) \frac{dY_{2b}^+(D_1)}{dy} Y_3^-(D_1) - \rho_2(D_1)\, Y_{2b}^+(D_1) \frac{dY_3^-(D_1)}{dy}\right\}$$

$$+ \left(\rho_4(D_2) \frac{dY_3^-(D_2)}{dy} Y_4^+(D_2) - \rho_3(D_2)\, Y_3^-(D_2) \frac{dY_4^+(D_2)}{dy}\right)$$

$$\times \left\{\rho_3(D_1) \frac{dY_{2b}^+(D_1)}{dy} Y_3^+(D_1) - \rho_2(D_1)\, Y_{2b}^+(D_1) \frac{dY_3^+(D_1)}{dy}\right\}\Big]$$

$$\times \Big[\left(Y_{2a}^+(y_o) \frac{dY_{2b}^-(y_o)}{dy} - \frac{dY_{2a}^+(y_o)}{dy} Y_{2b}^-(y_o)\right)$$

$$\times \left\{\rho_1(0)\, Y_1^-(0) \frac{dY_{2a}^-(0)}{dy} - \rho_2(0) \frac{dY_1^-(0)}{dy} Y_{2a}^-(0)\right\}$$

$$+\left(\frac{dY_{2a}^{-}(y_0)}{dy}Y_{2b}^{-}(y_0) - Y_{2a}^{-}(y_0)\frac{dY_{2b}^{-}(y_0)}{dy}\right)$$

$$\times\left\{\rho_1(0)\,Y_1^{-}(0)\frac{dY_{2a}^{+}(0)}{dy} - \rho_2(0)\frac{dY_1^{-}(0)}{dy}Y_{2a}^{+}(0)\right\}\Big]$$

$$-\Big[\Big(\rho_3(D_2)\,Y_3^{+}(D_2)\frac{dY_4^{+}(D_2)}{dy} - \rho_4(D_2)\frac{dY_3^{+}(D_2)}{dy}Y_4^{+}(D_2)\Big)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^{-}(D_1)}{dy}Y_3^{-}(D_1) - \rho_2(D_1)\,Y_{2b}^{-}(D_1)\frac{dY_3^{-}(D_1)}{dy}\right\}$$

$$+\left(\rho_4(D_2)\frac{dY_3^{-}(D_2)}{dy}Y_4^{+}(D_2) - \rho_3(D_2)\,Y_3^{-}(D_2)\frac{dY_4^{+}(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^{-}(D_1)}{dy}Y_3^{+}(D_1) - \rho_2(D_1)\,Y_{2b}^{-}(D_1)\frac{dY_3^{+}(D_1)}{dy}\right\}\Big]$$

$$\times\Big[\Big(Y_{2a}^{+}(y_0)\frac{dY_{2b}^{+}(y_0)}{dy} - \frac{dY_{2a}^{+}(y_0)}{dy}Y_{2b}^{+}(y_0)\Big)$$

$$\times\left\{\rho_1(0)\,Y_1^{-}(0)\frac{dY_{2a}^{-}(0)}{dy} - \rho_2(0)\frac{dY_1^{-}(0)}{dy}Y_{2a}^{-}(0)\right\}$$

$$+\left(\frac{dY_{2a}^{-}(y_o)}{dy}Y_{2b}^{+}(y_o) - Y_{2a}^{-}(y_o)\frac{dY_{2b}^{+}(y_o)}{dy}\right)$$

$$\times\left\{\rho_1(0)\,Y_1^{-}(0)\frac{dY_{2a}^{+}(0)}{dy} - \rho_2(0)\frac{dY_1^{-}(0)}{dy}Y_{2a}^{+}(0)\right\}\Big]\Big]\,. \qquad (364)$$

For constant speed of sound and ambieny density, (364) becomes

$$B_{1_c} = \frac{jk_r}{2\pi}\left(2\,\rho_2\,k_{y_2}\right)e^{-jk_{y_4}D_2}$$

$$\times\Big[\Big(\left(\rho_3\,k_{y_2} + \rho_2\,k_{y_3}\right)\left(\rho_4\,k_{y_3} - \rho_3\,k_{y_4}\right)e^{-jk_{y_2}D_1}\,e^{+jk_{y_2}y_o}\,e^{-jk_{y_3}D_2}\,e^{+jk_{y_3}D_1}$$

$$+\left(\rho_3\,k_{y_2} - \rho_2\,k_{y_3}\right)\left(\rho_3\,k_{y_4} + \rho_4\,k_{y_3}\right)e^{-jk_{y_2}D_1}\,e^{+jk_{y_2}y_o}\,e^{+jk_{y_3}D_2}\,e^{-jk_{y_3}D_1}$$

$$+\left(\rho_3\,k_{y_2} - \rho_2\,k_{y_3}\right)\left(\rho_4\,k_{y_3} - \rho_3\,k_{y_4}\right)e^{+jk_{y_2}D_1}\,e^{-jk_{y_2}y_o}\,e^{-jk_{y_3}D_2}\,e^{+jk_{y_3}D_1}$$

$$+\left(\rho_3 k_{y_2}+\rho_2 k_{y_3}\right)\left(\rho_3 k_{y_4}+\rho_4 k_{y_3}\right) e^{+jk_{y_2}D_1} e^{-jk_{y_2}y_0} e^{+jk_{y_3}D_2} e^{-jk_{y_3}D_1}\Big] /$$

$$\Big[2 k_{y_2} e^{-jk_{y_4}D_2}\Big[\left(\rho_1 k_{y_2}+\rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2}-\rho_2 k_{y_3}\right)$$

$$\times\left(\rho_4 k_{y_3}-\rho_3 k_{y_4}\right) e^{+jk_{y_2}D_1} e^{-jk_{y_3}D_2} e^{+jk_{y_3}D_1}$$

$$+\left(\rho_1 k_{y_2}+\rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2}+\rho_2 k_{y_3}\right)$$

$$\times\left(\rho_4 k_{y_3}+\rho_3 k_{y_4}\right) e^{+jk_{y_2}D_1} e^{+jk_{y_3}D_2} e^{-jk_{y_3}D_1}$$

$$-\left(\rho_1 k_{y_2}-\rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2}+\rho_2 k_{y_3}\right)$$

$$\times\left(\rho_4 k_{y_3}-\rho_3 k_{y_4}\right) e^{-jk_{y_2}D_1} e^{-jk_{y_3}D_2} e^{+jk_{y_3}D_1}$$

$$-\left(\rho_1 k_{y_2}-\rho_2 k_{y_1}\right)\left(\rho_3 k_{y_2}-\rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big]\Big] . \qquad (365)$$

Using the definitions of R_{21}, R_{23}, R_{34}, and T_{21} presented earlier, (365) may be reduced to

$$B_{1_c} = \frac{+jk_r}{4\pi k_{y_2}} T_{21}$$

$$\times \Big[e^{+jk_{y_2} y_0} \Big[R_{34} e^{-j2k_{y_2} D_1} e^{-j2k_{y_3} (D_2 - D_1)} + R_{23} e^{-j2k_{y_2} D_1} \Big] +$$

$$e^{-jk_{y_2} y_0} \Big[1 + R_{23} R_{34} e^{-j2k_{y_3} (D_2 - D_1)} \Big]\Big] /$$

$$\Big[R_{23} R_{34} e^{-j2k_{y_3} (D_2 - D_1)} + 1 - R_{21} R_{34} e^{-j2k_{y_3} (D_2 - D_1)} e^{-j2k_{y_2} D_1}$$

$$- R_{21} R_{23} e^{-j2k_{y_2} D_1} \Big] . \qquad (366)$$

Using conditions (328) and (329), (366) reduces to

$$B_{1_c} = \frac{+jk_r}{4\pi k_{y_2}} T_{21} \left[R_{23}\, e^{-j2k_{y_2} D}\, e^{+jk_{y_2} y_o} + e^{-jk_{y_2} y_o} \right] /$$

$$\left[1 - R_{21} R_{23}\, e^{-j2k_{y_2} D} \right]. \tag{367}$$

Equation (367) is equal to equation (268). Since (268) has already been verified, we can conclude that our derivation of the solutions for B_1, and hence, for B_{1_c} for the four media waveguide problem is correct.

For the unknown constant A_4, *Mathematica* provided the following numerator:

$$\begin{aligned}
num_8 = G_1 \big[& a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,7}\, a_{5,1}\, a_{6,4}\, a_{7,6} \\
& - a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,7}\, a_{5,1}\, a_{6,4}\, a_{7,6} + a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,7}\, a_{5,2}\, a_{6,4}\, a_{7,6} \\
& - a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,7}\, a_{5,3}\, a_{6,4}\, a_{7,6} - a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,7}\, a_{5,1}\, a_{6,5}\, a_{7,6} \\
& + a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,7}\, a_{5,1}\, a_{6,5}\, a_{7,6} - a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,7}\, a_{5,2}\, a_{6,5}\, a_{7,6} \\
& + a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,7}\, a_{5,3}\, a_{6,5}\, a_{7,6} - a_{1,3}\, a_{2,2}\, a_{3,5}\, a_{4,6}\, a_{5,1}\, a_{6,4}\, a_{7,7} \\
& + a_{1,2}\, a_{2,3}\, a_{3,5}\, a_{4,6}\, a_{5,1}\, a_{6,4}\, a_{7,7} - a_{1,1}\, a_{2,3}\, a_{3,5}\, a_{4,6}\, a_{5,2}\, a_{6,4}\, a_{7,7} \\
& + a_{1,1}\, a_{2,2}\, a_{3,5}\, a_{4,6}\, a_{5,3}\, a_{6,4}\, a_{7,7} + a_{1,3}\, a_{2,2}\, a_{3,4}\, a_{4,6}\, a_{5,1}\, a_{6,5}\, a_{7,7} \\
& - a_{1,2}\, a_{2,3}\, a_{3,4}\, a_{4,6}\, a_{5,1}\, a_{6,5}\, a_{7,7} + a_{1,1}\, a_{2,3}\, a_{3,4}\, a_{4,6}\, a_{5,2}\, a_{6,5}\, a_{7,7} \\
& - a_{1,1}\, a_{2,2}\, a_{3,4}\, a_{4,6}\, a_{5,3}\, a_{6,5}\, a_{7,7} \big].
\end{aligned} \tag{368}$$

Factoring (368) and collecting common terms yields the following *generic expression* for the *numerator* of A_4:

$$\mathrm{num}_8 = G_1\left(a_{4,7}\,a_{7,6} - a_{4,6}\,a_{7,7}\right)\left\{a_{3,4}\,a_{6,5} - a_{3,5}\,a_{6,4}\right\}$$
$$\times\left[a_{2,2}\left(a_{1,1}\,a_{5,3} - a_{1,3}\,a_{5,1}\right) - a_{2,3}\left(a_{1,1}\,a_{5,2} - a_{1,2}\,a_{5,1}\right)\right]. \qquad (369)$$

Substituting appropriate expressions into (369) and using (315) yields the following *general expression* for A_4:

$$A_4 = \frac{-k_r}{2\pi}\left(\rho_3(D_2)\frac{dY_3^+(D_2)}{dy}Y_3^-(D_2) - \rho_3(D_2)\,Y_3^+(D_2)\frac{dY_3^-(D_2)}{dy}\right)$$

$$\times\left\{\rho_2(D_1)\,Y_{2b}^+(D_1)\frac{dY_{2b}^-(D_1)}{dy} - \rho_2(D_1)\frac{dY_{2b}^+(D_1)}{dy}Y_{2b}^-(D_1)\right\}$$

$$\times\left[Y_{2a}^+(y_o)\left(\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^-(0) - \rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^-(0)}{dy}\right)\right.$$

$$\left. - Y_{2a}^-(y_o)\left(\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^+(0) - \rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^+(0)}{dy}\right)\right]\Big/$$

$$\left[\,\right|\left(\rho_3(D_2)\,Y_3^+(D_2)\frac{dY_4^+(D_2)}{dy} - \rho_4(D_2)\frac{dY_3^+(D_2)}{dy}Y_4^+(D_2)\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^+(D_1)}{dy}Y_3^-(D_1) - \rho_2(D_1)\,Y_{2b}^+(D_1)\frac{dY_3^-(D_1)}{dy}\right\}$$

$$+\left(\rho_4(D_2)\frac{dY_3^-(D_2)}{dy}Y_4^+(D_2)-\rho_3(D_2)\,Y_3^-(D_2)\frac{dY_4^+(D_2)}{dy}\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^+(D_1)}{dy}Y_3^+(D_1)-\rho_2(D_1)\,Y_{2b}^+(D_1)\frac{dY_3^+(D_1)}{dy}\right\}\Big]$$

$$\times\Big[\left(Y_{2a}^+(y_o)\frac{dY_{2b}^-(y_o)}{dy}-\frac{dY_{2a}^+(y_o)}{dy}Y_{2b}^-(y_o)\right)$$

$$\times\left\{\rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^-(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^-(0)\right\}$$

$$+\left(\frac{dY_{2a}^-(y_o)}{dy}Y_{2b}^-(y_o)-Y_{2a}^-(y_o)\frac{dY_{2b}^-(y_o)}{dy}\right)$$

$$\times\left\{\rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^+(0)}{dy}-\rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^+(0)\right\}\Big]$$

$$-\Big[\left(\rho_3(D_2)\,Y_3^+(D_2)\frac{dY_4^+(D_2)}{dy}-\rho_4(D_2)\frac{dY_3^+(D_2)}{dy}Y_4^+(D_2)\right)$$

$$\times\left\{\rho_3(D_1)\frac{dY_{2b}^-(D_1)}{dy}Y_3^-(D_1)-\rho_2(D_1)\,Y_{2b}^-(D_1)\frac{dY_3^-(D_1)}{dy}\right\}$$

$$
+ \left(\rho_4(D_2)\frac{dY_3^-(D_2)}{dy}Y_4^+(D_2) - \rho_3(D_2)\,Y_3^-(D_2)\frac{dY_4^+(D_2)}{dy}\right)
$$

$$
\times \left\{\rho_3(D_1)\frac{dY_{2b}^-(D_1)}{dy}Y_3^+(D_1) - \rho_2(D_1)\,Y_{2b}^-(D_1)\frac{dY_3^+(D_1)}{dy}\right\}\Big]
$$

$$
\times \Big[\left(Y_{2a}^+(y_o)\frac{dY_{2b}^+(y_o)}{dy} - \frac{dY_{2a}^+(y_o)}{dy}Y_{2b}^+(y_o)\right)
$$

$$
\times \left\{\rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^-(0)}{dy} - \rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^-(0)\right\}
$$

$$
+ \left(\frac{dY_{2a}^-(y_o)}{dy}Y_{2b}^+(y_o) - Y_{2a}^-(y_o)\frac{dY_{2b}^+(y_o)}{dy}\right)
$$

$$
\times \left\{\rho_1(0)\,Y_1^-(0)\frac{dY_{2a}^+(0)}{dy} - \rho_2(0)\frac{dY_1^-(0)}{dy}Y_{2a}^+(0)\right\}\Big]\Big]\,. \tag{370}
$$

For constant speed of sound and ambient density, (370) becomes

$$A_{4_c} = \frac{+jk_r}{2\pi} \left(2\,\rho_2\,k_{y_2}\right) \left(2\,\rho_3\,k_{y_3}\right)$$

$$\times \left[\left(\rho_1\,k_{y_2} - \rho_2\,k_{y_1}\right) e^{-jk_{y_2} y_0} + \left(\rho_1\,k_{y_2} + \rho_2\,k_{y_1}\right) e^{+jk_{y_2} y_0} \right] /$$

$$\Big[2\,k_{y_2}\; e^{-jk_{y_4} D_2} \Big[\left(\rho_1\,k_{y_2} + \rho_2\,k_{y_1}\right) \left(\rho_3\,k_{y_2} - \rho_2\,k_{y_3}\right)$$

$$\times \left(\rho_4\,k_{y_3} - \rho_3\,k_{y_4}\right) e^{+jk_{y_2} D_1}\; e^{-jk_{y_3} D_2}\; e^{+jk_{y_3} D_1}$$

$$+ \left(\rho_1\,k_{y_2} + \rho_2\,k_{y_1}\right) \left(\rho_3\,k_{y_2} + \rho_2\,k_{y_3}\right)$$

$$\times \left(\rho_4\,k_{y_3} + \rho_3\,k_{y_4}\right) e^{+jk_{y_2} D_1}\; e^{+jk_{y_3} D_2}\; e^{-jk_{y_3} D_1}$$

$$- \left(\rho_1\,k_{y_2} - \rho_2\,k_{y_1}\right) \left(\rho_3\,k_{y_2} + \rho_2\,k_{y_3}\right)$$

$$\times \left(\rho_4\,k_{y_3} - \rho_3\,k_{y_4}\right) e^{-jk_{y_2} D_1}\; e^{-jk_{y_3} D_2}\; e^{+jk_{y_3} D_1}$$

$$- \left(\rho_1 k_{y_2} - \rho_2 k_{y_1}\right) \left(\rho_3 k_{y_2} - \rho_2 k_{y_3}\right)$$

$$\times \left(\rho_4 k_{y_3} + \rho_3 k_{y_4}\right) e^{-jk_{y_2} D_1} e^{+jk_{y_3} D_2} e^{-jk_{y_3} D_1} \Big]\Big] . \qquad (371)$$

Using the definitions of R_{21}, R_{23}, R_{34}, and T_{23} presented earlier, and defining the transmission coefficient at the boundary between medium three and medium four as follows:

$$T_{34} = \frac{2 \rho_3 k_{y_3}}{\rho_4 k_{y_3} + \rho_3 k_{y_4}} , \qquad (372)$$

(371) may be reduced to

$$A_{4_c} = \frac{+jk_r}{4\pi k_{y_2}} T_{23} T_{34}$$

$$\times \Big[\Big[e^{+jk_{y_2} y_0} + R_{21} e^{-jk_{y_2} y_0} \Big] e^{-jk_{y_2} D_1} e^{-jk_{y_3} (D_2 - D_1)} e^{+jk_{y_4} D_2} \Big] /$$

$$\left[R_{23} R_{34} e^{-j2k_{y_3}(D_2 - D_1)} + 1 - R_{21} R_{34} e^{-j2k_{y_2} D_1} e^{-j2k_{y_3}(D_2 - D_1)}\right.$$

$$\left. - R_{21} R_{23} e^{-j2k_{y_2} D_1}\right] . \tag{373}$$

Using conditions (328), (329), and letting

$$T_{34} = 1 , \tag{374}$$

and, as a result,

$$k_{y_4} = k_{y_3} , \tag{375}$$

(373) reduces to

$$A_{4_c} = \frac{+jk_r}{4\pi k_{y_2}} T_{23} \left[e^{+jk_{y_2} y_o} + R_{21} e^{-jk_{y_2} y_o}\right] e^{-jk_{y_2} D} \quad e^{+jk_{y_3} D} \quad /$$

$$\left[1 - R_{21} R_{23} e^{-j2k_{y_2} D}\right] . \tag{376}$$

Equation (376) is equal to both equations (275) and (355). Since (275) has already been verified, we can conclude that our derivation of the

solutions for A_4, and hence, for A_{4_c} for the four media waveguide problem is correct.

To summarize the results for the four media waveguide with plane, parallel boundaries, for the general case, in which speed of sound and density are arbitrary functions of depth, the unknown constants are given by (322), (333), (340), (346), (352), (358), (364), and (370). As in the three media waveguide case, the denominators in these expressions are exactly the same. For constant speed of sound and density, the unknown constants are given by (327), (335), (342), (348), (354), (360), (366), and (373).

Having completed two successful tests of the programming technique, we now return our attention to the general waveguide problem. As in the simpler examples, the compact vector-matrix system equation (198) applies. For the general waveguide problem, the vector matrix quantities involved are as follows:

- $\mathbf{A}$ is the 28 by 17 matrix of coefficients,

$$
\begin{aligned}
\mathbf{A} = \; & [[0\ a_{1,2}\ a_{1,3}\ a_{1,4}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \\
& [a_{2,1}\ 0\ 0\ 0\ a_{2,5}\ a_{2,6}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \\
& [0\ 0\ a_{3,3}\ a_{3,4}\ 0\ 0\ 0\ a_{3,8}\ a_{3,9}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \\
& [0\ 0\ 0\ 0\ a_{4,5}\ a_{4,6}\ 0\ 0\ 0\ a_{4,10}\ a_{4,11}\ 0\ 0\ 0\ 0\ 0\ 0] \\
& [0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{5,8}\ a_{5,9}\ 0\ 0\ a_{5,12}\ a_{5,13}\ 0\ 0\ 0\ 0] \\
& [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{6,10}\ a_{6,11}\ 0\ 0\ a_{6,14}\ a_{6,15}\ 0\ 0] \\
& [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{7,12}\ a_{7,13}\ 0\ 0\ a_{7,16}\ 0] \\
& [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{8,14}\ a_{8,15}\ 0\ a_{8,17}] \\
& [0\ a_{9,2}\ a_{9,3}\ a_{9,4}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \\
& [a_{10,1}\ 0\ 0\ 0\ a_{10,5}\ a_{10,6}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \\
& [0\ a_{11,2}\ a_{11,3}\ a_{11,4}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]
\end{aligned}
$$

$[a_{12,1}\ 0\ 0\ 0\ a_{12,5}\ a_{12,6}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$
$[a_{13,1}\ 0\ 0\ 0\ a_{13,5}\ a_{13,6}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$
$[0\ a_{14,2}\ a_{14,3}\ a_{14,4}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{15,8}\ a_{15,9}\ 0\ 0\ a_{15,12}\ a_{15,13}\ 0\ 0\ 0\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{16,10}\ a_{16,11}\ 0\ 0\ a_{16,14}\ a_{16,15}\ 0\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{17,8}\ a_{17,9}\ 0\ 0\ a_{17,12}\ a_{17,13}\ 0\ 0\ 0\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{18,10}\ a_{18,11}\ 0\ 0\ a_{18,14}\ a_{18,15}\ 0\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{19,10}\ a_{19,11}\ 0\ 0\ a_{19,14}\ a_{19,15}\ 0\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{20,8}\ a_{20,9}\ 0\ 0\ a_{20,12}\ a_{20,13}\ 0\ 0\ 0\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{21,12}\ a_{21,13}\ 0\ 0\ a_{21,16}\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{22,14}\ a_{22,15}\ 0\ a_{22,17}]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{23,12}\ a_{23,13}\ 0\ 0\ a_{23,16}\ 0]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{24,14}\ a_{24,15}\ 0\ a_{24,17}]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{25,14}\ a_{25,15}\ 0\ a_{25,17}]$
$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a_{26,12}\ a_{26,13}\ 0\ 0\ a_{26,16}\ 0]$
$[0\ 0\ a_{27,3}\ a_{27,4}\ 0\ 0\ 0\ a_{27,8}\ a_{27,9}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$
$[0\ 0\ 0\ 0\ a_{28,5}\ a_{28,6}\ -1\ 0\ 0\ a_{28,10}\ a_{28,11}\ 0\ 0\ 0\ 0\ 0\ 0]]$

where

$$a_{1,2} = -\rho_1(y_S)\, J_n(k_{r_1} r)\, Y_1^-(y_S) \tag{377}$$

$$a_{1,3} = \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^+(y_S) \tag{378}$$

$$a_{1,4} = \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^-(y_S) \tag{379}$$

$$a_{2,1} = -\rho_1(y_S)\, J_n(k_{r_1} r)\, Y_1^-(y_S) \tag{380}$$

$$a_{2,5} = \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^+(y_S) \tag{381}$$

$$a_{2,6} = \rho_2(y_S)\, J_n(k_{r_2} r)\, Y_{2a}^-(y_S) \tag{382}$$

$$a_{3,3} = Y_{2a}^+(y_o) \tag{383}$$

$$a_{3,4} = Y_{2a}^{-}(y_o) \tag{384}$$

$$a_{3,8} = - Y_{2b}^{+}(y_o) \tag{385}$$

$$a_{3,9} = - Y_{2b}^{-}(y_o) \tag{386}$$

$$a_{4,5} = Y_{2a}^{+}(y_o) \tag{387}$$

$$a_{4,6} = Y_{2a}^{-}(y_o) \tag{388}$$

$$a_{4,10} = - Y_{2b}^{+}(y_o) \tag{389}$$

$$a_{4,11} = - Y_{2b}^{-}(y_o) \tag{390}$$

$$a_{5,8} = \rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^{+}(y_{B_1}) \tag{391}$$

$$a_{5,9} = \rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^{-}(y_{B_1}) \tag{392}$$

$$a_{5,12} = - \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^{+}(y_{B_1}) \tag{393}$$

$$a_{5,13} = - \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^{-}(y_{B_1}) \tag{394}$$

$$a_{6,10} = \rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^{+}(y_{B_1}) \tag{395}$$

$$a_{6,11} = \rho_2(y_{B_1})\, J_n(k_{r_2} r)\, Y_{2b}^{-}(y_{B_1}) \tag{396}$$

$$a_{6,14} = - \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^{+}(y_{B_1}) \tag{397}$$

$$a_{6,15} = - \rho_3(y_{B_1})\, J_n(k_{r_3} r)\, Y_3^{-}(y_{B_1}) \tag{398}$$

$$a_{7,12} = \rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^{+}(y_{B_2}) \tag{399}$$

$$a_{7,13} = \rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_2}) \tag{400}$$

$$a_{7,16} = -\rho_4(y_{B_2})\, J_n(k_{r_4} r)\, Y_4^+(y_{B_2}) \tag{401}$$

$$a_{8,14} = \rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^+(y_{B_2}) \tag{402}$$

$$a_{8,15} = \rho_3(y_{B_2})\, J_n(k_{r_3} r)\, Y_3^-(y_{B_2}) \tag{403}$$

$$a_{8,17} = -\rho_4(y_{B_2})\, J_n(k_{r_4} r)\, Y_4^+(y_{B_2}) \tag{404}$$

$$a_{9,2} = -J_n(k_{r_1} r)\, \frac{dY_1^-(y_S)}{dy} \tag{405}$$

$$a_{9,3} = J_n(k_{r_2} r)\, \frac{dY_{2a}^+(y_S)}{dy} \tag{406}$$

$$a_{9,4} = J_n(k_{r_2} r)\, \frac{dY_{2a}^-(y_S)}{dy} \tag{407}$$

$$a_{10,1} = -J_n(k_{r_1} r)\, \frac{dY_1^-(y_S)}{dy} \tag{408}$$

$$a_{10,5} = J_n(k_{r_2} r)\, \frac{dY_{2a}^+(y_S)}{dy} \tag{409}$$

$$a_{10,6} = J_n(k_{r_2} r)\, \frac{dY_{2a}^-(y_S)}{dy} \tag{410}$$

$$a_{11,2} = -k_{r_1}\, \frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)}\, Y_1^-(y_S) \tag{411}$$

$$a_{11,3} = k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2a}^{+}(y_S) \tag{412}$$

$$a_{11,4} = k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2a}^{-}(y_S) \tag{413}$$

$$a_{12,1} = - J_n(k_{r_1} r)\, Y_1^{-}(y_S) \tag{414}$$

$$a_{12,5} = J_n(k_{r_2} r)\, Y_{2a}^{+}(y_S) \tag{415}$$

$$a_{12,6} = J_n(k_{r_2} r)\, Y_{2a}^{-}(y_S) \tag{416}$$

$$a_{13,1} = - k_{r_1} \frac{dJ_n(k_{r_1} r)}{d(k_{r_1} r)} Y_1^{-}(y_S) \tag{417}$$

$$a_{13,5} = k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2a}^{+}(y_S) \tag{418}$$

$$a_{13,6} = k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2a}^{-}(y_S) \tag{419}$$

$$a_{14,2} = - J_n(k_{r_1} r)\, Y_1^{-}(y_S) \tag{420}$$

$$a_{14,3} = J_n(k_{r_2} r)\, Y_{2a}^{+}(y_S) \tag{421}$$

$$a_{14,4} = J_n(k_{r_2} r)\, Y_{2a}^{-}(y_S) \tag{422}$$

$$a_{15,8} = J_n(k_{r_2} r) \frac{dY_{2b}^{+}(y_{B_1})}{dy} \tag{423}$$

$$a_{15,9} = J_n(k_{r_2} r) \frac{dY_{2b}^{-}(y_{B_1})}{dy} \tag{424}$$

$$a_{15,12} = -\, J_n(k_{r_3} r) \frac{dY_{3}^{+}(y_{B_1})}{dy} \tag{425}$$

$$a_{15,13} = -\, J_n(k_{r_3} r) \frac{dY_{3}^{-}(y_{B_1})}{dy} \tag{426}$$

$$a_{16,10} = J_n(k_{r_2} r) \frac{dY_{2b}^{+}(y_{B_1})}{dy} \tag{427}$$

$$a_{16,11} = J_n(k_{r_2} r) \frac{dY_{2b}^{-}(y_{B_1})}{dy} \tag{428}$$

$$a_{16,14} = -\, J_n(k_{r_3} r) \frac{dY_{3}^{+}(y_{B_1})}{dy} \tag{429}$$

$$a_{16,15} = -\, J_n(k_{r_3} r) \frac{dY_{3}^{-}(y_{B_1})}{dy} \tag{430}$$

$$a_{17,8} = k_{r_2} \frac{dJ_n(k_{r_2}\, r)}{d(k_{r_2}\, r)} Y_{2b}^{+}(y_{B_1}) \tag{431}$$

$$a_{17,9} = k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2b}^{-}(y_{B_1}) \tag{432}$$

$$a_{17,12} = - k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^{+}(y_{B_1}) \tag{433}$$

$$a_{17,13} = - k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^{-}(y_{B_1}) \tag{434}$$

$$a_{18,10} = J_n(k_{r_2} r)\, Y_{2b}^{+}(y_{B_1}) \tag{435}$$

$$a_{18,11} = J_n(k_{r_2} r)\, Y_{2b}^{-}(y_{B_1}) \tag{436}$$

$$a_{18,14} = - J_n(k_{r_3} r)\, Y_3^{+}(y_{B_1}) \tag{437}$$

$$a_{18,15} = - J_n(k_{r_3} r)\, Y_3^{-}(y_{B_1}) \tag{438}$$

$$a_{19,10} = k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2b}^{+}(y_{B_1}) \tag{439}$$

$$a_{19,11} = k_{r_2} \frac{dJ_n(k_{r_2} r)}{d(k_{r_2} r)} Y_{2b}^{-}(y_{B_1}) \tag{440}$$

$$a_{19,14} = -k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^+(y_{B_1}) \tag{441}$$

$$a_{19,15} = -k_{r_3} \frac{dJ_n(k_{r_3} r)}{d(k_{r_3} r)} Y_3^-(y_{B_1}) \tag{442}$$

$$a_{20,8} = J_n(k_{r_2} r)\, Y_{2b}^+(y_{B_1}) \tag{443}$$

$$a_{20,9} = J_n(k_{r_2} r)\, Y_{2b}^-(y_{B_1}) \tag{444}$$

$$a_{20,12} = -J_n(k_{r_3} r)\, Y_3^+(y_{B_1}) \tag{445}$$

$$a_{20,13} = -J_n(k_{r_3} r)\, Y_3^-(y_{B_1}) \tag{446}$$

$$a_{21,12} = J_n(k_{r_3} r) \frac{dY_3^+(y_{B_2})}{dy} \tag{447}$$

$$a_{21,13} = J_n(k_{r_3} r) \frac{dY_3^-(y_{B_2})}{dy} \tag{448}$$

$$a_{21,16} = -J_n(k_{r_4} r) \frac{dY_4^+(y_{B_2})}{dy} \tag{449}$$

$$a_{22,14} = J_n(k_{r_3} r) \frac{dY_3^+(y_{B_2})}{dy} \tag{450}$$

$$a_{22,15} = J_n(k_{r_3} r) \frac{dY_3^-(y_{B_2})}{dy} \tag{451}$$

$$a_{22,17} = -J_n(k_{r_4} r)\frac{dY_4^+(y_{B_2})}{dy} \tag{452}$$

$$a_{23,12} = k_{r_3}\frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)}Y_3^+(y_{B_2}) \tag{453}$$

$$a_{23,13} = k_{r_3}\frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)}Y_3^-(y_{B_2}) \tag{454}$$

$$a_{23,16} = -k_{r_4}\frac{dJ_n(k_{r_4}\, r)}{d(k_{r_4}\, r)}Y_4^+(y_{B_2}) \tag{455}$$

$$a_{24,14} = J_n(k_{r_3} r)\, Y_3^+(y_{B_2}) \tag{456}$$

$$a_{24,15} = J_n(k_{r_3} r)\, Y_3^-(y_{B_2}) \tag{457}$$

$$a_{24,17} = -J_n(k_{r_4} r)\, Y_4^+(y_{B_2}) \tag{458}$$

$$a_{25,14} = k_{r_3}\frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)}Y_3^+(y_{B_2}) \tag{459}$$

$$a_{25,15} = k_{r_3}\frac{dJ_n(k_{r_3}\, r)}{d(k_{r_3}\, r)}Y_3^-(y_{B_2}) \tag{460}$$

$$a_{25,17} = - k_{r_4} \frac{dJ_n(k_{r_4} r)}{d(k_{r_4} r)} Y_4^+(y_{B_2}) \tag{461}$$

$$a_{26,12} = J_n(k_{r_3} r)\, Y_3^+(y_{B_2}) \tag{462}$$

$$a_{26,13} = J_n(k_{r_3} r)\, Y_3^-(y_{B_2}) \tag{463}$$

$$a_{26,16} = - J_n(k_{r_4} r)\, Y_4^+(y_{B_2}) \tag{464}$$

$$a_{27,3} = \frac{dY_{2a}^+(y_o)}{dy} \tag{465}$$

$$a_{27,4} = \frac{dY_{2a}^-(y_o)}{dy} \tag{466}$$

$$a_{27,8} = - \frac{dY_{2b}^+(y_o)}{dy} \tag{467}$$

$$a_{27,9} = - \frac{dY_{2b}^-(y_o)}{dy} \tag{468}$$

$$a_{28,5} = \frac{dY_{2a}^+(y_o)}{dy} \tag{469}$$

$$a_{28,6} = \frac{dY_{2a}^-(y_o)}{dy} \tag{470}$$

$$a_{28,10} = - \frac{dY_{2b}^+(y_o)}{dy} \tag{471}$$

$$a_{28,11} = - \frac{dY_{2b}^-(y_o)}{dy} \tag{472}$$

and

• $\mathbf{x}$ is the 17 by one vector of unknown constants,

$$\mathbf{x} = \{A_1\, B_1\, A_{2a}\, B_{2a}\, C_{2a}\, D_{2a}\, G_2\, A_{2b}\, B_{2b}\, C_{2b}\, D_{2b}\, A_3\, B_3\, C_3\, D_3\, A_4\, B_4\}^T, \quad (473)$$

where the superscript T indicates the transpose matrix operator (which means that $\mathbf{x}$ is a column vector).

The vector of known constants, $\mathbf{b}$, is simply made up of the right-hand sides of (149) through (176), or more specifically

$$\mathbf{b} = \{0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ G_1\ 0\}^T, \quad (474)$$

where here again the transpose operator is used to indicate that $\mathbf{b}$ is a column vector.

One should note that the arbitrary constant G_1 appears in the known constant vector while the arbitrary constant G_2 appears in the unknown constant vector. This occurs because G_1 represents the known amount of discontinuity required to achieve the free-space Green's function solution under the necessary conditions for that solution to exist (i.e., constant speed of sound, no boundaries). G_2, on the other hand, is really an artifact of the method used to derive these boundary condition equations, and, as such, should be treated as an unknown quantity in the general case.

In the general waveguide problem, the form of matrix $\mathbf{A}$ leads to some complications. First, the fact that the number of equations (i.e., 28) is greater than the number of unknowns (i.e., 17) implies that the solution to this problem will not be unique. Secondly, the fact that the matrix $\mathbf{A}$ is not a

square matrix implies that the simple matrix inversion technique of (222) cannot be utilized. Thus, an alternate solution technique must be employed to set up the problem so that *Mathematica* may be used in the solution.

Gelb (1974) describes this situation (more equations than unknowns) as an overdetermined case. Gelb and Haykin (1986) both suggest the use of a pseudoinverse matrix in obtaining a least squares estimate for the vector $\mathbf{x}$. The pseudoinverse matrix is defined by Gelb as follows for real matrices $\mathbf{A}$:

$$\mathbf{A}^{\#} = (\mathbf{A}^{T}\,\mathbf{A})^{-1}\,\mathbf{A}^{T}. \tag{475}$$

where $\mathbf{A}^{\#}$ is the 17 by 28 pseudoinverse matrix.

Haykin and Menke (1984, p. 253) both define a similar pseudoinverse matrix for complex matrices $\mathbf{A}$ as follows:

$$\mathbf{A}^{\#} = (\mathbf{A}^{H}\,\mathbf{A})^{-1}\,\mathbf{A}^{H}. \tag{476}$$

where the superscript H indicates the Hermetian or complex conjugate transpose matrix operator.

In the solution for the general waveguide case, we will use a combination of these pseudoinverse matrix techniques. This combination incorporates a 28 by 28 weighting matrix, $\mathbf{W}$, which allows us to obtain a weighted least squares estimate for the vector $\mathbf{x}$. Therefore, our approach will be to use the weighted pseudoinverse formulation suggested by Gelb and Menke (1984, p. 54) with the complex conjugate transpose operators

suggested by Haykin in lieu of the standard (and less general) transpose operator. Thus, the pseudoinverse matrix to be used in this thesis is defined as follows:

$$\mathbf{A}^{\#} = (\mathbf{A}^{H} \mathbf{W} \mathbf{A})^{-1} \mathbf{A}^{H} \mathbf{W} . \tag{477}$$

Using this pseudoinverse matrix, a weighted least squares estimate for the vector $\mathbf{x}$ may be calculated as follows:

$$\mathbf{x} = \mathbf{A}^{\#} \mathbf{b} . \tag{478}$$

The following *Mathematica* code was developed to solve the general waveguide problem using this weighted least squares technique:

```
b = {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 0, 0, 0, G1, 0};

w = {{w1c1, w1c2, w1c3, w1c4, w1c5, w1c6, w1c7, w1c8,
      w1c9, w1c10, w1c11, w1c12, w1c13, w1c14, w1c15,
      w1c16, w1c17, w1c18, w1c19, w1c20, w1c21, w1c22,
      w1c23, w1c24, w1c25, w1c26, w1c27, w1c28},
     {w2c1, w2c2, w2c3, w2c4, w2c5, w2c6, w2c7, w2c8,
      w2c9, w2c10, w2c11, w2c12, w2c13, w2c14, w2c15,
      w2c16, w2c17, w2c18, w2c19, w2c20, w2c21, w2c22,
      w2c23, w2c24, w2c25, w2c26, w2c27, w2c28},
     {w3c1, w3c2, w3c3, w3c4, w3c5, w3c6, w3c7, w3c8,
      w3c9, w3c10, w3c11, w3c12, w3c13, w3c14, w3c15,
      w3c16, w3c17, w3c18, w3c19, w3c20, w3c21, w3c22,
      w3c23, w3c24, w3c25, w3c26, w3c27, w3c28},
     {w4c1, w4c2, w4c3, w4c4, w4c5, w4c6, w4c7, w4c8,
      w4c9, w4c10, w4c11, w4c12, w4c13, w4c14, w4c15,
      w4c16, w4c17, w4c18, w4c19, w4c20, w4c21, w4c22,
      w4c23, w4c24, w4c25, w4c26, w4c27, w4c28},
     {w5c1, w5c2, w5c3, w5c4, w5c5, w5c6, w5c7, w5c8,
      w5c9, w5c10, w5c11, w5c12, w5c13, w5c14, w5c15,
      w5c16, w5c17, w5c18, w5c19, w5c20, w5c21, w5c22,
```

w5c23, w5c24, w5c25, w5c26, w5c27, w5c28},
{w6c1, w6c2, w6c3, w6c4, w6c5, w6c6, w6c7, w6c8,
w6c9, w6c10, w6c11, w6c12, w6c13, w6c14, w6c15,
w6c16, w6c17, w6c18, w6c19, w6c20, w6c21, w6c22,
w6c23, w6c24, w6c25, w6c26, w6c27, w6c28},
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w7c9, w7c10, w7c11, w7c12, w7c13, w7c14, w7c15,
w7c16, w7c17, w7c18, w7c19, w7c20, w7c21, w7c22,
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w8c9, w8c10, w8c11, w8c12, w8c13, w8c14, w8c15,
w8c16, w8c17, w8c18, w8c19, w8c20, w8c21, w8c22,
w8c23, w8c24, w8c25, w8c26, w8c27, w8c28},
{w9c1, w9c2, w9c3, w9c4, w9c5, w9c6, w9c7, w9c8,
w9c9, w9c10, w9c11, w9c12, w9c13, w9c14, w9c15,
w9c16, w9c17, w9c18, w9c19, w9c20, w9c21, w9c22,
w9c23, w9c24, w9c25, w9c26, w9c27, w9c28},
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w10c23, w10c24, w10c25, w10c26, w10c27, w10c28},
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w11c9, w11c10, w11c11, w11c12, w11c13, w11c14, w11c15,
w11c16, w11c17, w11c18, w11c19, w11c20, w11c21, w11c22,
w11c23, w11c24, w11c25, w11c26, w11c27, w11c28},
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w12c9, w12c10, w12c11, w12c12, w12c13, w12c14, w12c15,
w12c16, w12c17, w12c18, w12c19, w12c20, w12c21, w12c22,
w12c23, w12c24, w12c25, w12c26, w12c27, w12c28},
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w13c9, w13c10, w13c11, w13c12, w13c13, w13c14, w13c15,
w13c16, w13c17, w13c18, w13c19, w13c20, w13c21, w13c22,
w13c23, w13c24, w13c25, w13c26, w13c27, w13c28},
{w14c1, w14c2, w14c3, w14c4, w14c5, w14c6, w14c7, w14c8,
w14c9, w14c10, w14c11, w14c12, w14c13, w14c14, w14c15,
w14c16, w14c17, w14c18, w14c19, w14c20, w14c21, w14c22,
w14c23, w14c24, w14c25, w14c26, w14c27, w14c28},
{w15c1, w15c2, w15c3, w15c4, w15c5, w15c6, w15c7, w15c8,
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w15c16, w15c17, w15c18, w15c19, w15c20, w15c21, w15c22, w15c23, w15c24, w15c25, w15c26, w15c27, w15c28},
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        w28c23, w28c24, w28c25, w28c26, w28c27, w28c28}};

a = {{0, a1c2, a1c3, a1c4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {a2c1, 0, 0, 0, a2c5, a2c6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {0, 0, a3c3, a3c4, 0, 0, 0, a3c8, a3c9, 0, 0, 0, 0, 0, 0, 0, 0},
     {0, 0, 0, 0, a4c5, a4c6, 0, 0, 0, a4c10, a4c11, 0, 0, 0, 0, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, a5c8, a5c9, 0, 0, a5c12, a5c13, 0, 0, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, a6c10, a6c11, 0, 0, a6c14, a6c15, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a7c12, a7c13, 0, 0, a7c16, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a8c14, a8c15, 0, a8c17}
     {0, a9c2, a9c3, a9c4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {a10c1, 0, 0, 0, a10c5, a10c6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {0, a11c2, a11c3, a11c4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {a12c1, 0, 0, 0, a12c5, a12c6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {a13c1, 0, 0, 0, a13c5, a13c6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {0, a14c2, a14c3, a14c4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, a15c8, a15c9, 0, 0, a15c12, a15c13, 0, 0, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, a16c10, a16c11, 0, 0, a16c14, a16c15, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, a17c8, a17c9, 0, 0, a17c12, a17c13, 0, 0, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, a18c10, a18c11, 0, 0, a18c14, a18c15, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, a19c10, a19c11, 0, 0, a19c14, a19c15, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, a20c8, a20c9, 0, 0, a20c12, a20c13, 0, 0, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a21c12, a21c13, 0, 0, a21c16, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a22c14, a22c15, 0, a22c17},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a23c12, a23c13, 0, 0, a23c16, 0},
```

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{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a24c14, a24c15, 0, a24c17},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a25c14, a25c15, 0, a25c17},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a26c12, a26c13, 0, 0, a26c16, 0},
{0, 0, a27c3, a27c4, 0, 0, 0, a27c8, a27c9, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, a28c5, a28c6, -1, 0, 0, a28c10, a28c11, 0, 0, 0, 0, 0, 0}};

aherm = Transpose[Conjugate[a]];

x = (Inverse[aherm.w.a]).aherm.w.b
```

Again, the lower case w was required to represent the weighting matrix **W** due to the *Mathematica* variable naming convention mentioned earlier (i.e., matrices have variable names which begin with lower case letters since *Mathematica* reserves variable names which begin with capital letters for built-in functions). The variable "aherm" represents the complex conjugate transpose of the matrix **A** (i.e., $\mathbf{A}^{\mathbf{H}}$).

This code ran on a Macintosh II computer which is equipped with five megabytes (MB) of random access memory (RAM) for about 25 minutes before halting due to an "out of memory" error. This same code was run on a different Macintosh II computer equipped with similar hardware and a software package which allowed access of up to eight MB of hard disk space for use as virtual memory. Thus, *Mathematica* had 13 MB of RAM available to it. On this 13 MB machine, the code ran for about 75 minutes before it again halted on an "out of memory" error. Before attempting an alternate approach, the code was run in steps to see which calculation was causing the trouble. The multiplication of $\mathbf{A}^{\mathbf{H}}$,**W**, and **A** ran successfully as did the multiplication of $\mathbf{A}^{\mathbf{H}}$, **W**, and **b**. The problem recurred when an attempt was

made to take the inverse of the product of $\mathbf{A}^{\mathbf{H}}$,$\mathbf{W}$, and $\mathbf{A}$. We speculate that the problem occurs because the program is required to store and operate on a large number of string variables, thereby requiring large quantities of memory to store intermediate results. To confirm this, we attempted to take the inverse of a 17 by 17 symbolic matrix using only simple generic variable names (such as j1c1) without success. We must conclude that *Mathematica* requires too much memory to successfully run this type of symbolic problem on a personal computer. The methodology should be validated when less memory intensive or main frame based symbolic programs become available.

In order to attempt the use of *Mathematica*'s LinearSolve function, we must first modify the matrix $\mathbf{A}$ so that it is a square matrix. This may be accomplished while maintaining the integrity of our weighted least squares formulation as follows:

Recall that

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}\,. \tag{198}$$

Premultiplying both sides of (198) by the matrix $\mathbf{W}$ yields

$$\mathbf{W}\,\mathbf{A}\,\mathbf{x} = \mathbf{W}\,\mathbf{b}\,. \tag{479}$$

Premultiplying both sides of (479) by the matrix $\mathbf{A}^{\mathbf{H}}$ yields

$$\mathbf{A}^{\mathbf{H}}\,\mathbf{W}\,\mathbf{A}\,\mathbf{x} = \mathbf{A}^{\mathbf{H}}\,\mathbf{W}\,\mathbf{b}\,. \tag{480}$$

If we let

$$\mathbf{j} = \mathbf{A}^{H}\,\mathbf{W}\,\mathbf{A}\,, \tag{481}$$

and

$$\mathbf{d} = \mathbf{A}^{H}\,\mathbf{W}\,\mathbf{b}\,, \tag{482}$$

then we have reformulated the problem as desired to

$$\mathbf{j}\,\mathbf{x} = \mathbf{d}\;, \tag{483}$$

where $\mathbf{j}$ is the 17 by 17 matrix defined by (481) and $\mathbf{d}$ is the 17 by 1 column vector defined by (482).

The following revised *Mathematica* code was developed for this reformulated problem:

```
b = {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0, 0, G1, 0};

w = {{w1c1, w1c2, w1c3, w1c4, w1c5, w1c6, w1c7, w1c8,
      w1c9, w1c10, w1c11, w1c12, w1c13, w1c14, w1c15,
      w1c16, w1c17, w1c18, w1c19, w1c20, w1c21, w1c22,
      w1c23, w1c24, w1c25, w1c26, w1c27, w1c28},
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      w3c16, w3c17, w3c18, w3c19, w3c20, w3c21, w3c22,
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      w4c16, w4c17, w4c18, w4c19, w4c20, w4c21, w4c22,
      w4c23, w4c24, w4c25, w4c26, w4c27, w4c28},
```

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 w26c16, w26c17, w26c18, w26c19, w26c20, w26c21, w26c22,
 w26c23, w26c24, w26c25, w26c26, w26c27, w26c28},
{w27c1, w27c2, w27c3, w27c4, w27c5, w27c6, w27c7, w27c8,
 w27c9, w27c10, w27c11, w27c12, w27c13, w27c14, w27c15,
 w27c16, w27c17, w27c18, w27c19, w27c20, w27c21, w27c22,
 w27c23, w27c24, w27c25, w27c26, w27c27, w27c28},
{w28c1, w28c2, w28c3, w28c4, w28c5, w28c6, w28c7, w28c8,
 w28c9, w28c10, w28c11, w28c12, w28c13, w28c14, w28c15,
 w28c16, w28c17, w28c18, w28c19, w28c20, w28c21, w28c22,
 w28c23, w28c24, w28c25, w28c26, w28c27, w28c28}};

a - {{0, a1c2, a1c3, a1c4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {a2c1, 0, 0, 0, a2c5, a2c6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, a3c3, a3c4, 0, 0, 0, a3c8, a3c9, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, a4c5, a4c6, 0, 0, 0, a4c10, a4c11, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, a5c8, a5c9, 0, 0, a5c12, a5c13, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, a6c10, a6c11, 0, 0, a6c14, a6c15, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a7c12, a7c13, 0, 0, a7c16, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a8c14, a8c15, 0, a8c17},
    {0, a9c2, a9c3, a9c4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {a10c1, 0, 0, 0, a10c5, a10c6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, a11c2, a11c3, a11c4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {a12c1, 0, 0, 0, a12c5, a12c6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {a13c1, 0, 0, 0, a13c5, a13c6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, a14c2, a14c3, a14c4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, a15c8, a15c9, 0, 0, a15c12, a15c13, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, a16c10, a16c11, 0, 0, a16c14, a16c15, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, a17c8, a17c9, 0, 0, a17c12, a17c13, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, a18c10, a18c11, 0, 0, a18c14, a18c15, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, a19c10, a19c11, 0, 0, a19c14, a19c15, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, a20c8, a20c9, 0, 0, a20c12, a20c13, 0, 0, 0, 0},
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a21c12, a21c13, 0, 0, a21c16, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a22c14, a22c15, 0, a22c17},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a23c12, a23c13, 0, 0, a23c16, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a24c14, a24c15, 0, a24c17},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a25c14, a25c15, 0, a25c17},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a26c12, a26c13, 0, 0, a26c16, 0},
{0, 0, a27c3, a27c4, 0, 0, 0, a27c8, a27c9, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, a28c5, a28c6, -1, 0, 0, a28c10, a28c11, 0, 0, 0, 0, 0, 0}};

aherm = Transpose[Conjugate[a]];

j = aherm.w.a;

d = aherm.w.b;

LinearSolve[j,d]
```

This revised code ran on the five MB Macintosh for about 25 minutes before halting on an "out of memory" error. It also ran on the 13 MB Macintosh for about 110 minutes before halting on the memory error. These failures have led us to conclude that the solution of the general problem is possible with this technique but is not practical with currently available hardware/software configurations.

One additional test was run using the code generated for the general case employing *Mathematica* 's LinearSolve function. In this test case, the three media waveguide with plane, parallel boundaries was simulated by setting the weighting matrix, **W**, equal to the identity matrix (using the *Mathematica* command: w = IdentityMatrix[28];), setting appropriate values of the **A** matix equal to zero, and directly assigning values to the $\mathbf{A}^{\mathbf{H}}$ matrix as follows (in *Mathematica* code):

```
aherm = {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {a1c2CC, 0, 0, 0, 0, 0, 0, 0, a9c2CC, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
         {a1c3CC, 0, a3c3CC, 0, 0, 0, 0, 0, a9c3CC, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a27c3CC, 0},
         {a1c4CC, 0, a3c4CC, 0, 0, 0, 0, 0, a9c4CC, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a27c4CC, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {0, 0, a3c8CC, 0, a5c8CC, 0, 0, 0, 0, 0, 0, 0, 0, 0, a15c8CC,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a27c8CC, 0},
         {0, 0, a3c9CC, 0, a5c9CC, 0, 0, 0, 0, 0, 0, 0, 0, 0, a15c9CC,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a27c9CC, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, a5c12CC, 0, 0, 0, 0, 0, 0, 0, 0, a15c12CC, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0}},
```

where the notation CC in each of the variable names indicates that the element represents the complex conjugate of the appropriate element of the **A** matrix.

This code ran on the five MB Macintosh for ten days without converging to a solution. This additional failure confirms our suspicions that the general problem is not practically solved using current technology.

V. CONCLUSIONS AND RECOMMENDATIONS

The main purpose of this thesis was to obtain the symbolic solution of a multilayer (four fluid media) ocean waveguide problem. The waveguide was assumed to have depth-dependent ambient density and sound-speed profiles in all fluid media, and arbitrarily shaped boundaries between all fluid media. A system of 28 equations in 17 unknowns was generated by satisfying all of the boundary conditions (including the boundary condition at the source) in cylindrical coordinates. A weighted least squares estimation technique was employed to formulate a symbolic solution to this overdetermined (more equations than unknowns) case. A computer program capable of performing symbolic algebra was sought to minimize the number of assumptions required to be made, thereby maximizing the generality of the solution obtained. *Mathematica* (version 1.2.1 f33 (enhanced)) for the Macintosh computer was selected for this work due to its availability at the Naval Postgraduate School and its symbolic algebra capabilities. *Mathematica* code was developed which programmed the weighted least squares estimation technique for the most general case. Unfortunately, this code was unable to provide a solution to the most general case due to software and hardware limitations (i.e., speed and random access memory problems).

By relaxing the arbitrary boundary shape assumption, *Mathematica* code was developed which programmed a direct solution to the three media waveguide problem for plane, parallel boundaries. This code ran successfully, and provided results which could be verified by direct

comparison with the known solution to this classical (i.e., three fluid media with plane, parallel boundaries and constant ambient density and sound speed in each medium) waveguide problem. During this process, built-in *Mathematica* functions were used in an attempt to simplify the resulting symbolic expressions. This effort revealed that these built-in functions lacked sufficient sophistication for applications of this complexity. This lack of sophisication resulted in manual reduction of the program output so that verification was possible. *Mathematica* code was then developed to solve the four media waveguide problem for plane, parallel boundaries. This code also ran successfully and yielded results which could be verified using known classical waveguide solutions when some judicious assumptions were made to mathematically eliminate the fourth medium.

In addition to validating the symbolic solution technique, this thesis provides a series of generic expressions for the unknown constants for each of the three and four media waveguide problems with plane, parallel boundaries. Each of the generic expressions is a combination of generic variables whose definitions are provided in the text. Each of these generic variables can be programmed in a high level language (i.e., FORTRAN) as a unique subprogram or function. In this manner, the unknown constants can be calculated by combinations of calls to appropriate subroutines. This modular programming technique is enhanced by the fact that each of the generic expressions has a common denominator, which can also be programmed in a similar manner.

The generic results from the four media waveguide problem for plane, parallel boundaries should be programmed in FORTRAN. When this programming has been completed, the standard test cases (i.e., three fluid media with plane, parallel boundaries and constant ambient density and sound speed in each medium) should be run on the new code to verify the results. This verification will provide additional credibility to *Mathematica* 's output.

It is recommended that the most general case (i.e., four fluid media with arbitrarily shaped boundaries) be attempted again when one of the following conditions are met:

- a Macintosh computer with more than 13MB of RAM becomes available at the Naval Postgraduate School,
- an advanced version of Mathematica is released, or
- another computer program with symbolic algebra capabilities becomes available for use on a workstation or main frame computer.

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